



TITLE:

BV analysis of tachyon fluctuation around multi-brane solutions in cubic string field theory

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CITATION:

Hata, Hiroyuki. BV analysis of tachyon fluctuation around multi-brane solutions in cubic string field theory. Journal of High Energy Physics 2016, 2016: 22.

ISSUE DATE:

2016-05

URL:

<http://hdl.handle.net/2433/216031>

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PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: December 22, 2015

REVISED: March 24, 2016

ACCEPTED: April 24, 2016

PUBLISHED: May 4, 2016

BV analysis of tachyon fluctuation around multi-brane solutions in cubic string field theory

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ABSTRACT: We study whether the tachyon mode exists as a physical fluctuation on the 2-brane solution and on the tachyon vacuum solution in cubic open string field theory. Our analysis is based on the Batalin-Vilkovisky formalism. We first construct a set of six string states which corresponds to the set of fields and anti-fields containing the tachyon field. Whether the tachyon field can exist as a physical fluctuation is determined by the 6×6 matrix defining the anti-bracket in the present sector. If the matrix is degenerate/non-degenerate, the tachyon field is physical/unphysical. Calculations for the pure-gauge type solutions in the framework of the KBc algebra and using the K_ϵ -regularization lead to the expected results. Namely, the matrix for the anti-bracket is degenerate/non-degenerate in the case of the 2-brane/tachyon-vacuum solution. Our analysis is not complete, in particular, in that we have not identified the four-fold degeneracy of tachyon fluctuation on the 2-brane solution, and moreover that the present six states do not satisfy the hermiticity condition.

KEYWORDS: BRST Quantization, D-branes, String Field Theory, Tachyon Condensation

ARXIV EPRINT: [1511.04187](https://arxiv.org/abs/1511.04187)

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1 Introduction

After the discovery of exact tachyon vacuum solution [1] in cubic string field theory (CSFT) followed by its concise understanding [2] in terms of the KBc algebra [3], there have been considerable developments in the construction of multi-brane solutions [4–7]. The identification of a solution as the n -brane one representing n pieces of D25-branes has been done from its energy density consideration. However, for the complete identification, we have to show that the physical excitations on the solution are those of the open string and, in particular, that each excitation has n^2 degeneracy. For the tachyon vacuum solution ($n = 0$), a general proof has been given for the absence of physical excitations [8]. On the other hand, for n -brane solution with $n \geq 2$, no formal existence proof nor an explicit construction of the excitations has been given.¹

In this paper, we present an explicit analysis of fluctuations around multi-brane solutions in the framework of the Batalin-Vilkovisky (BV) formalism [10, 11]. Our analysis is not a complete one, but is rather a first step toward the final understanding. First, our analysis is restricted only to the tachyon vacuum solution and the 2-brane one. Second, we do not solve the general excitation modes on the solution. Our analysis is restricted to the tachyon mode among all the excitations.

Let us explain our analysis in more detail. We are interested in the kinetic term of the action of CSFT expanded around a multi-brane solution:

$$\mathcal{S}_0 = \frac{1}{2} \int \Phi * \mathcal{Q}\Phi, \quad (1.1)$$

where \mathcal{Q} is the BRST operator in the background of the solution, and Φ is the fluctuation around the solution. Previous arguments have been mainly on the presence of the homotopy operator \mathcal{A} on the tachyon vacuum solution satisfying $\mathcal{Q}\mathcal{A} = \mathcal{I}$ with \mathcal{I} being the identity string field. If there exists a well-defined \mathcal{A} , it implies that there are no physical excitations at all. In this paper, we carry out a different kind of analysis. We consider a candidate tachyon field $\chi(x)$ as a fluctuation around a class of multi-brane solutions, and examine whether χ represents a genuine physical excitation or it is unphysical. In the former case, the lagrangian of χ contained in (1.1) should be the ordinary one:²

$$\mathcal{L}_\chi = -\frac{1}{2} \left((\partial_\mu \chi)^2 + m^2 \chi^2 \right). \quad (1.2)$$

On the other hand, if χ is unphysical, it should be a member of unphysical BRST quartet fields $(\chi, C, \bar{C}, \mathcal{B})$ with the lagrangian given by a BRST-exact form [12]:

$$\mathcal{L}_{\text{quartet}} = i\delta_B \left[\bar{C} \left((\square - m^2) \chi - \frac{1}{2} \mathcal{B} \right) \right] = -\mathcal{B} (\square - m^2) \chi + \frac{1}{2} \mathcal{B}^2 - i\bar{C} (\square - m^2) C, \quad (1.3)$$

where the BRST transformation δ_B (satisfying the nilpotency $\delta_B^2 = 0$) is defined by

$$\delta_B \chi = C, \quad \delta_B C = 0, \quad \delta_B \bar{C} = i\mathcal{B}, \quad \delta_B \mathcal{B} = 0. \quad (1.4)$$

¹See [9], for a construction of multi-brane solutions and the fluctuation modes around them by introducing the boundary condition changing operators.

²The space-time metric used in this paper is the mostly plus one; $g_{\mu\nu} = \text{diag}(-1, 1, 1, \dots, 1)$.

In CSFT which has been constructed in the BV formalism, the lagrangian for unphysical χ is not of the type (1.3) containing the auxiliary field \mathcal{B} , but is rather the one obtained by integrating out \mathcal{B} :

$$\mathcal{L}'_{\text{quartet}} = -\frac{1}{2} [(\square - m^2) \chi]^2 - i\bar{C} (\square - m^2) C. \quad (1.5)$$

This is invariant under the redefined BRST transformation δ'_B :

$$\delta'_B \chi = C, \quad \delta'_B C = 0, \quad \delta'_B \bar{C} = i(\square - m^2) \chi. \quad (1.6)$$

Note that δ'_B is nilpotent only on-shell; in particular, we have $(\delta'_B)^2 \bar{C} = i(\square - m^2) C$.

Our analysis is carried out within the framework of the BV formalism. We first construct six first-quantized string states $u_i(k)$ carrying center-of-mass momentum k_μ . These six states correspond to the three fields (χ, C, \bar{C}) in (1.5) as well as their anti-fields $(\chi_\star, C_\star, \bar{C}_\star)$. We call the six states u_i the tachyon BV states. Then, we examine the 6×6 matrix $\omega_{ij} = \int u_i u_j$ given by the CSFT integration. In fact, this ω_{ij} is the matrix defining the anti-bracket in the BV formalism, and it determines whether χ is physical or unphysical. If ω_{ij} is non-degenerate, namely, $\det \omega_{ij} \neq 0$, χ is unphysical. More precisely, after the gauge-fixing by removing the anti-fields, we obtain the lagrangian (1.5) of an unphysical system. On the other hand, if ω_{ij} is degenerate, it implies that the six states u_i are not independent, and therefore, some of the fields/anti-fields necessary for making χ unphysical are missing. Concrete analysis shows that the lagrangian for χ in the case of degenerate ω_{ij} is the physical one (1.2).

We consider multi-brane solutions in CSFT given formally as the pure-gauge $UQ_B U^{-1}$ with U specified by a function $G(K)$ of K in the KBc -algebra (see (3.2)). The point is that the eigenvalues of K are in the range $K \geq 0$, and various physical quantities associated with the solution such as the energy density are not well-defined due to the singularity at $K = 0$. Therefore, we introduce the K_ε -regularization of replacing K in $UQ_B U^{-1}$ by $K_\varepsilon = K + \varepsilon$ with ε being a positive infinitesimal [5]. Then, the regularized solution $(UQ_B U^{-1})_{K \rightarrow K_\varepsilon}$ is no longer exactly pure-gauge, and the zero or the pole of $G(K)$ at $K = 0$ is interpreted as the origin of the non-trivial energy density of the apparently pure-gauge configuration [4–7].³ Namely, ε from the infinitesimal violation of pure-gauge is enhanced by $1/\varepsilon$ from the singularity at $K = 0$ to lead to non-trivial results for the solution.

This phenomenon of $\varepsilon \times (1/\varepsilon)$ giving non-trivial results also occurs in ω_{ij} in our BV analysis. By the gauge transformation which transforms $UQ_B U^{-1}$ to zero, the regularized solution $(UQ_B U^{-1})_{K \rightarrow K_\varepsilon}$ is transformed to an apparently $O(\varepsilon)$ quantity. Then, the corresponding BRST operator \mathcal{Q} is almost equal to the original Q_B ; $\mathcal{Q} = Q_B + O(\varepsilon)$. Therefore, the matrix ω_{ij} for the six tachyon BV states is reduced to a degenerate one if we simply put $\varepsilon = 0$ without taking into account the singularity at $K = 0$. Namely, there exists a physical tachyon field on any n -brane solution of the pure-gauge type in the naive analysis. The total absence of physical excitations expected on the tachyon vacuum should rather

³The zero and pole of $G(K)$ at $K = \infty$ also make the pure-gauge solutions non-trivial and more rich [7]. However, we do not consider this type of solutions in this paper.

be a non-trivial phenomenon coming from $\varepsilon \times (1/\varepsilon) \neq 0$. Our interest here is whether this phenomenon does not occur on n -branes with $n \geq 2$.

In CSFT, the meaning of the EOM, $Q_B \Psi + \Psi^2 = 0$, is not so simple. When we consider whether the EOM is satisfied by a candidate solution Ψ_S , we have to specify the test string field Ψ_T and examine whether the EOM test, $\int \Psi_T * (Q_B \Psi_S + \Psi_S^2) = 0$, holds or not. It is in general impossible that the EOM test holds for any Ψ_T , and the EOM test restricts both the solution and the fluctuations around it. For the pure-gauge type solutions mentioned above, the EOM against itself (namely, $\Psi_T = \Psi_S$) is satisfied only for the tachyon vacuum solution and the 2-brane one (and, of course, for the single-brane solution $\Psi_S = 0$) [5]. The correct value of the energy density can also be reproduced only for these two solutions. Therefore, in this paper, we carry out calculations of ω_{ij} for these two kinds of solutions with $n = 0$ and 2. Then, we need to take into account the EOM also in the construction of the tachyon BV states u_i on each solution. For the BV analysis, the EOM must hold against the commutator $\Psi_T = [u_i, u_j]$ as well as $\Psi_T = u_i$ themselves, and this is in fact a non-trivial problem, in particular, for the 2-brane solution. For devising such u_i , we multiply the naive expression of u_0 with the lowest ghost number by the functions of K_ε , $L(K_\varepsilon)$ and $1/R(K_\varepsilon)$, from the left and the right, respectively, and define the whole set of six u_i by the operation of \mathcal{Q} . Then, we obtain the constraints on $L(K_\varepsilon)$ and $R(K_\varepsilon)$ from the requirement of the EOM. The existence of $L(K_\varepsilon)$ and $R(K_\varepsilon)$ also affects the calculation of ω_{ij} .

There is another important technical point in our BV analysis. The matrix $\omega_{ij} = \int u_i u_j$ and the EOM test against the commutator $\Psi_T = [u_i, u_j]$ are functions of k^2 of the momentum k_μ carried by u_i . Then, a problem arises: some of these quantities contain terms depending on ε of the K_ε -regularization in a manner such as $\varepsilon^{\min(2k^2-1,1)}$, which diverges in the limit $\varepsilon \rightarrow 0$ for a smaller k^2 and tends to zero for a larger k^2 . Therefore, we *define* them as the “analytic continuation” from the region of sufficiently large k^2 (namely, sufficiently space-like k_μ) to drop this type of ε -dependent terms.

Next, we comment on the “cohomology approach” to the problem of physical fluctuation around a multi-brane solution. In this approach, we consider the BRST cohomology $\text{Ker } \mathcal{Q} / \text{Im } \mathcal{Q}$, namely, we solve $\mathcal{Q}u_1(k) = 0$ for $u_1(k)$ which carries ghost number one and is not \mathcal{Q} -exact. However, the meaning of (non-)equality in $\mathcal{Q}u_1 = 0$ and $u_1 \neq \mathcal{Q}(\ast)$ is subtle for multi-brane solutions of the pure-gauge type discussed in this paper due to the singularity at $K = 0$. To make these equations precise, we should introduce the K_ε -regularization and consider their inner-products (CSFT integrations) with states in the space of fluctuations. We would like to stress that our BV analysis indeed gives information for solving the BRST cohomology problem within the K_ε -regularization. (The present BV analysis can identify some of the non-trivial elements of $\text{Ker } \mathcal{Q} / \text{Im } \mathcal{Q}$. However, it cannot give the complete answer to the cohomology problem since we consider only a set of trial BV states.) We will explain the interpretation of our results of the BV analysis in the context of the cohomology approach in sections 4.4 and 5.2. We also comment that the analysis of the BRST cohomology around the tachyon vacuum by evaluating the kinetic term of the action of the fluctuation in the level truncation approximation [13–15] has some relevance to the present BV approach.

Then, finally in the Introduction, we state our results obtained in this paper. For the tachyon vacuum solution, we find that the matrix ω_{ij} is non-degenerate. This implies that our candidate tachyon field is an unphysical one belonging to a BRST quartet. On the other hand, for the 2-brane solution, ω_{ij} turns out to be degenerate, implying that the tachyon field is a physical one. These results are both what we expect for each solution. However, we have not succeeded in identifying the whole of the 2^2 tachyon fields which should exist on the 2-brane solution. In addition, the six tachyon BV states in this paper have a problem that they do not satisfy the hermiticity requirement (see section 3.5).

The organization of the rest of this paper is as follows. In section 2, we recapitulate the BV formalism used in this paper, and give examples of the BV states on the unstable vacuum. In section 3, we present the construction of the six tachyon BV states on a generic pure-gauge type solution, and prepare various formulas necessary for the BV analysis. In section 4, we carry out the calculation of the EOM against u_i and $[u_i, u_j]$ and of each component of ω_{ij} on the 2-brane solution to confirm the existence of a physical tachyon field. Next, in section 5, we repeat the same analysis for the tachyon vacuum solution. There we find that the candidate tachyon field is unphysical. We summarize the paper and discuss future problems in section 6. In the appendices, we present various technical details used in the text.

2 BV formalism for CSFT

The action of CSFT on the unstable vacuum [16],⁴

$$S[\Psi] = \int \left(\frac{1}{2} \Psi * Q_B \Psi + \frac{1}{3} \Psi^3 \right), \quad (2.1)$$

satisfies the BV equation:

$$\int \left(\frac{\delta S}{\delta \Psi} \right)^2 = 0. \quad (2.2)$$

Concretely, we have

$$\frac{\delta S}{\delta \Psi} = Q_B \Psi + \Psi^2, \quad (2.3)$$

and the BV equation holds due to (i) the nilpotency $Q_B^2 = 0$ of the BRST operator Q_B , (ii) the derivation property of Q_B on the $*$ -product, (iii) the property $\int Q_B(\cdots) = 0$, (iv) the associativity of the $*$ -product, and (v) the cyclicity $\int A_1 * A_2 = (-1)^{A_1 A_2} \int A_2 * A_1$ valid for any two string fields A_1 and A_2 .⁵ The BV equation is a basic requirement in the construction of gauge theories including SFT. The BV equation implies the gauge invariance of the action. Moreover, it gives a consistent way of gauge-fixing and quantization of the theory.

In this paper, we are interested in CSFT expanded around a non-trivial solution Ψ_S satisfying the EOM:

$$Q_B \Psi_S + \Psi_S^2 = 0. \quad (2.4)$$

⁴We have put the open string coupling constant equal to one.

⁵ $(-1)^A = +1$ (-1) when A is Grassmann-even (-odd).

Expressing the original string field Ψ in (2.1) as

$$\Psi = \Psi_S + \Phi, \quad (2.5)$$

with Φ being the fluctuation, we obtain

$$S[\Psi] = S[\Psi_S] + \int \Phi * (Q_B \Psi_S + \Psi_S^2) + \mathcal{S}_{\Psi_S}[\Phi]. \quad (2.6)$$

The second term on the r.h.s. of (2.6) should vanish due to the EOM (2.4). However, for multi-brane solutions in CSFT, this EOM term cannot vanish for all kinds of fluctuations Φ as stated in the Introduction. This is the case even for the tachyon vacuum solution. In this paper, we restrict the fluctuation Φ around Ψ_S to those for which the EOM term of (2.6) vanishes. We will see later that the EOM term must also vanish against the commutator among the fluctuations.

The last term of (2.6) is the action of the fluctuation:

$$\mathcal{S}_{\Psi_S}[\Phi] = \int \left(\frac{1}{2} \Phi * \mathcal{Q}_{\Psi_S} \Phi + \frac{1}{3} \Phi^3 \right). \quad (2.7)$$

The only difference between the two actions (2.1) and (2.7) is that the BRST operator Q_B in the former is replaced with \mathcal{Q}_{Ψ_S} , the BRST operator around the solution Ψ_S . The operation of \mathcal{Q}_{Ψ_S} on any string field A with a generic ghost number is defined by

$$\mathcal{Q}_{\Psi_S} A = Q_B A + \Psi_S * A - (-1)^A A * \Psi_S. \quad (2.8)$$

The BV equation for \mathcal{S}_{Ψ_S} ,

$$\int \left(\frac{\delta \mathcal{S}_{\Psi_S}}{\delta \Phi} \right)^2 = 0, \quad (2.9)$$

which is formally equivalent to (2.2) for the original S , also holds since \mathcal{Q}_{Ψ_S} satisfies the same three basic properties as Q_B does; (i), (ii) and (iii) mentioned below (2.3). Among them, the nilpotency $\mathcal{Q}_{\Psi_S}^2 = 0$ is a consequence of the EOM; namely, we have from (2.8)

$$\mathcal{Q}_{\Psi_S}^2 A = [Q_B \Psi_S + \Psi_S^2, A]. \quad (2.10)$$

On the other hand, the other two properties (ii) and (iii) hold for any Ψ_S irrespectively of whether it satisfies the EOM or not. In the following, we omit the subscript Ψ_S in \mathcal{S}_{Ψ_S} unless necessary.

2.1 BV equation in terms of component fields

Here, we consider the BV equation (2.9) for the action (2.7) in terms of the component fields.⁶ Let $\{u_i(k)\}$ be a “complete set” of states of fluctuation around Ψ_S (here, we take as Ψ_S a translationally invariant solution, and k_μ is the center-of-mass momentum of the

⁶See, for example, [17, 18] for the BV formalism for a general supermanifold of fields and anti-fields. The matrix ω_{ij} in [18] corresponds to $(-1)^{\varphi_i} \omega_{ij}$ in this paper.

fluctuation). Note that each $u_i(k)$ is a string field. Then, we expand the fluctuation field Φ as

$$\Phi = \int_k \sum_i u_i(k) \varphi^i(k), \quad (2.11)$$

where $\varphi^i(k)$ is the component field corresponding to the state $u_i(k)$, and \int_k is short for $\int d^{26}k/(2\pi)^{26}$. In (2.11), $u_i(k)$ may carry any ghost number $N_{\text{gh}}(u_i)$, and the ghost number of the corresponding φ^i must satisfy

$$N_{\text{gh}}(u_i) + N_{\text{gh}}(\varphi^i) = N_{\text{gh}}(\Phi) = 1. \quad (2.12)$$

Then, we define the matrix $\omega_{ij}(k)$ and its inverse $\omega^{ij}(k)$ by

$$\int u_i(k') u_j(k) = \omega_{ij}(k) \times (2\pi)^{26} \delta^{26}(k' + k), \quad (2.13)$$

and

$$\sum_j \omega^{ij}(k) \omega_{jk}(k) = \delta^i_k. \quad (2.14)$$

Here, we are assuming that ω_{ij} is *non-degenerate*, namely, that the inverse matrix ω^{ij} exists.⁷ In particular, the number of the basis $u_i(k)$ must be even. Note that ω_{ij} and ω^{ij} are non-vanishing only for (i, j) satisfying $N_{\text{gh}}(u_i) + N_{\text{gh}}(u_j) = 3$, and therefore,

$$N_{\text{gh}}(\varphi^i) + N_{\text{gh}}(\varphi^j) = -1. \quad (2.15)$$

Note also that these matrices are symmetric in the following sense:

$$\omega_{ij}(k) = \omega_{ji}(-k), \quad \omega^{ij}(k) = \omega^{ji}(-k). \quad (2.16)$$

The completeness relation of the set $\{u_i\}$ reads

$$A = \int_k \sum_{i,j} u_i(k) \omega^{ij}(k) \int u_j(-k) * A, \quad (2.17)$$

for any string field A , and hence we have⁸

$$\frac{\delta}{\delta\Phi} = \int_k \sum_{i,j} u_i(k) \omega^{ij}(k) (-1)^{\varphi^j} \frac{\delta}{\delta\varphi^j(-k)}. \quad (2.18)$$

Using (2.18) in (2.9), we obtain the BV equation in terms of the component fields:

$$\int_k \sum_{i,j} \omega^{ij}(k) \frac{\delta\mathcal{S}}{\delta\varphi^i(k)} \frac{\delta\mathcal{S}}{\delta\varphi^j(-k)} = 0. \quad (2.19)$$

⁷Precisely speaking, our assumption here is that $\det \omega_{ij}(k)$ is not identically equal to zero as a function of k_μ . ω_{ij} being degenerate at some points in the k_μ space is allowed.

⁸The sign factor $(-1)^{\varphi^j}$ in (2.18) is due to the fact that the CSFT integration \int is Grassmann-odd. In this paper, $\delta/\delta\varphi^j$ for a Grassmann-odd φ^j is defined to be the left-derivative $\overrightarrow{\delta}/\delta\varphi^j$.

It is convenient to take the Darboux basis where the matrix $\omega_{ij}(k)$ takes the following form:

$$\omega_{ij}(k) = \begin{pmatrix} 0 & D(-k) \\ D(k) & 0 \end{pmatrix}, \quad D(k) = \text{diag}(a^1(k), a^2(k), \dots). \quad (2.20)$$

Denoting the corresponding component fields, namely, the pair of fields and anti-fields, as $\{\phi^i(k), \phi_\star^i(k)\}$ with the index i running only half of that for $\{\varphi^i\}$, the BV equation (2.19) reads

$$\int_k \sum_i a^i(k)^{-1} \frac{\delta \mathcal{S}}{\delta \phi_\star^i(-k)} \frac{\delta \mathcal{S}}{\delta \phi^i(k)} = 0. \quad (2.21)$$

Then, the gauge-fixed action $\widehat{\mathcal{S}}$ and the BRST transformation $\widehat{\delta}_B$ under which $\widehat{\mathcal{S}}$ is invariant are given by

$$\widehat{\mathcal{S}}[\phi] = \mathcal{S}|_L, \quad \widehat{\delta}_B \phi^i = i a^i(k)^{-1} \frac{\delta \mathcal{S}}{\delta \phi_\star^i(-k)} \Big|_L, \quad (2.22)$$

where $|_L$ denotes the restriction to the Lagrangian submanifold defined by the gauge-fermion $\Upsilon[\phi]$:

$$L : \quad \phi_\star^i = \frac{\delta \Upsilon[\phi]}{\delta \phi^i}. \quad (2.23)$$

The simplest choice for Υ is of course $\Upsilon = 0$.

2.2 Examples of BV basis on the unstable vacuum

For CSFT on the unstable vacuum, the BV basis $\{u_i(k)\}$ consists of an infinite number of first quantized string states of all ghost numbers. Though the whole BV basis is infinite dimensional, we can consider a subbasis with non-degenerate ω_{ij} and consisting of a finite number of states which are connected by the operation of Q_B and are orthogonal (in the sense of $\omega_{ij} = 0$) to any states outside the subbasis.

Here, we present two examples of BV subbasis with non-degenerate ω_{ij} . For our later purpose, we present them using the KBc algebra in the sliver frame. The KBc algebra and the correlators in the sliver frame are summarized in appendix A. In the rest of this paper, we omit “sub” for the BV subbasis and simply write “BV basis” since we will not consider the full BV basis.

2.2.1 Unphysical BV basis of photon longitudinal mode

Our first example is the unphysical BV basis associated with the longitudinal mode of the photon on the unstable vacuum $\Psi_S = 0$. Namely, we consider the unphysical model obtained by restricting the photon field to the pure-gauge, $A_\mu(x) = \partial_\mu \chi(x)$. The corre-

sponding BV basis consists of the following six states:

$$\begin{aligned}
 N_{\text{gh}}=0 : \quad u_0(k) &= \frac{1}{\sqrt{2}} e^{-\alpha K} V_k e^{-\alpha K}, \\
 N_{\text{gh}}=1 : \quad u_{1A}(k) &= \frac{i}{\sqrt{2}} e^{-\alpha K} c[K, V_k] e^{-\alpha K}, & u_{1B}(k) &= \frac{-i}{\sqrt{2}} e^{-\alpha K} [K, c] V_k e^{-\alpha K}, \\
 N_{\text{gh}}=2 : \quad u_{2A}(k) &= \frac{i}{\sqrt{2}} e^{-\alpha K} c[K, c] [K, V_k] e^{-\alpha K}, & u_{2B}(k) &= \frac{i}{\sqrt{2}} e^{-\alpha K} c[K, [K, c]] V_k e^{-\alpha K}, \\
 N_{\text{gh}}=3 : \quad u_3(k) &= \frac{1}{\sqrt{2}} e^{-\alpha K} c[K, [K, c]] [K, c] V_k e^{-\alpha K},
 \end{aligned} \tag{2.24}$$

where α is a constant,

$$\alpha = \frac{\pi}{4}, \tag{2.25}$$

and V_k is the vertex operator of momentum k_μ at the origin:

$$V_k = e^{ik_\mu X^\mu(0,0)}. \tag{2.26}$$

These six states $u_i(k)$ are all chosen to be hermitian in the sense that

$$u_i(k)^\dagger = u_i(-k). \tag{2.27}$$

Among the six u_i , u_{1A} is the photon state with longitudinal polarization k_μ . The operation of Q_B on the six states (2.24) is given as follows:

$$\begin{aligned}
 iQ_B u_0(k) &= u_{1A}(k) - k^2 u_{1B}(k), \\
 Q_B \begin{pmatrix} u_{1A} \\ u_{1B} \end{pmatrix} &= - \begin{pmatrix} k^2 \\ 1 \end{pmatrix} (u_{2A}(k) + u_{2B}(k)), \\
 iQ_B \begin{pmatrix} u_{2A}(k) \\ u_{2B}(k) \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} k^2 u_3(k), \\
 Q_B u_3(k) &= 0.
 \end{aligned} \tag{2.28}$$

The non-trivial components of the 6×6 matrix $\omega_{ij}(k)$ are given by

$$\omega_{0,3} = -1, \quad \begin{pmatrix} \omega_{1A,2A} & \omega_{1A,2B} \\ \omega_{1B,2A} & \omega_{1B,2B} \end{pmatrix} = \begin{pmatrix} k^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.29}$$

and therefore ω_{ij} is non-degenerate.⁹ Moreover, the present basis $\{u_i\}$ is already Darboux as seen from (2.29). Then, expanding the string field Ψ as

$$\Psi = \int_k \left\{ u_0(k) C(k) + u_{1A}(k) \chi(k) + u_{1B}(k) \bar{C}_\star(k) + u_{2A}(k) \chi_\star(k) + u_{2B}(k) \bar{C}(k) + u_3(k) C_\star(k) \right\}, \tag{2.30}$$

⁹Though $\omega_{ij}(k)$ is degenerate at $k^2 = 0$, this is not a problem as we mentioned in footnote 7.

and using (2.28) and (2.29), we find that the kinetic term of the CSFT action (2.1) is given by

$$\begin{aligned} S_0 &= \frac{1}{2} \int \Psi * Q_B \Psi \\ &= \int_k \left\{ -\frac{1}{2} (k^2 \chi(-k) + \bar{C}_*(-k)) (k^2 \chi(k) + \bar{C}_*(k)) + ik^2 (\bar{C}(-k) - \chi_*(-k)) C(k) \right\}. \end{aligned} \quad (2.31)$$

Finally, the gauge-fixed action \hat{S}_0 and the BRST transformation $\hat{\delta}_B$ in the gauge $\bar{C}_* = \chi_* = C_* = 0$ are given using (2.22) by

$$\hat{S}_0 = \int_k \left\{ -\frac{1}{2} k^2 \chi(-k) k^2 \chi(k) + ik^2 \bar{C}(-k) C(k) \right\}, \quad (2.32)$$

and

$$\hat{\delta}_B \chi(k) = C(k), \quad \hat{\delta}_B \bar{C}(k) = -ik^2 \chi(k), \quad \hat{\delta}_B C(k) = 0. \quad (2.33)$$

This is the $m^2 = 0$ version of the unphysical system given in (1.5) and (1.6).

2.2.2 BV basis of the tachyon mode

Our second example is the BV basis for the tachyon mode on the unstable vacuum. It consists only of two states: the tachyon state u_1 and its BRST-transform u_2 :

$$N_{\text{gh}} = 1 : u_1(k) = e^{-\alpha K} c V_k e^{-\alpha K}, \quad N_{\text{gh}} = 2 : u_2(k) = e^{-\alpha K} c K c V_k e^{-\alpha K}, \quad (2.34)$$

with

$$Q_B u_1(k) = -(k^2 - 1) u_2(k), \quad Q_B u_2(k) = 0. \quad (2.35)$$

The 2×2 matrix ω_{ij} is non-degenerate since we have

$$\omega_{1,2}(k) = 1. \quad (2.36)$$

Expressing the string field as

$$\Psi = \int_k (u_1(k) \phi(k) + u_2(k) \phi_*(k)), \quad (2.37)$$

the kinetic term reads

$$S_0 = -\frac{1}{2} \int_k \phi(-k) (k^2 - 1) \phi(k), \quad (2.38)$$

which does not contain the anti-field ϕ_* . The gauge-fixed action \hat{S}_0 is the same as S_0 , the ordinary kinetic term of the tachyon field ϕ . The BRST transformation of ϕ is of course equal to zero; $\hat{\delta}_B \phi = i \delta S_0 / \delta \phi_*|_L = 0$.

3 Tachyon BV states around a multi-brane solution

We consider the fluctuation around a multi-brane solution Ψ_ε given as the K_ε -regularization of the pure-gauge UQ_BU^{-1} [5]:

$$\Psi_\varepsilon = (UQ_BU^{-1})_\varepsilon = c \frac{K_\varepsilon}{G_\varepsilon} Bc (1 - G_\varepsilon), \quad (3.1)$$

where U and its inverse U^{-1} are specified by a function $G(K)$ of K :

$$U = 1 - Bc(1 - G(K)), \quad U^{-1} = 1 + \frac{1}{G(K)} Bc(1 - G(K)). \quad (3.2)$$

Here and in the following, \mathcal{O}_ε for a quantity \mathcal{O} containing K denotes the K_ε -regularized one; $\mathcal{O}_\varepsilon = \mathcal{O}|_{K \rightarrow K_\varepsilon = K + \varepsilon}$. Therefore, we have $G_\varepsilon \equiv G(K_\varepsilon)$ in (3.1). Although the EOM is satisfied automatically by the pure-gauge UQ_BU^{-1} , the K_ε -regularization breaks the EOM by the $O(\varepsilon)$ term:

$$Q_B\Psi_\varepsilon + \Psi_\varepsilon * \Psi_\varepsilon = \varepsilon \times c \frac{K_\varepsilon}{G_\varepsilon} c (1 - G_\varepsilon). \quad (3.3)$$

As we saw in [5], this $O(\varepsilon)$ breaking of the EOM can be enhanced by the singularity at $K = 0$ to lead to non-trivial results for the EOM against Ψ_ε itself:

$$\int \Psi_\varepsilon * (Q_B\Psi_\varepsilon + \Psi_\varepsilon^2) = \varepsilon \times \int Bc G_\varepsilon c \frac{K_\varepsilon}{G_\varepsilon} c G_\varepsilon c \frac{K_\varepsilon}{G_\varepsilon}. \quad (3.4)$$

We found that (3.4) vanishes for $G(K)$ having a simple zero, a simple pole or none at all at $K = 0$, which we expect to represent the tachyon vacuum, the 2-brane and the 1-brane, respectively, from their energy density values. For $G(K)$ with higher order zero or pole at $K = 0$, (3.4) becomes non-vanishing. Therefore, in this paper, we consider the following two $G(K)$ as concrete examples:

$$G_{\text{tv}}(K) = \frac{K}{1 + K}, \quad G_{\text{2b}}(K) = \frac{1 + K}{K}, \quad (3.5)$$

which correspond to the tachyon vacuum and the 2-brane, respectively.

For our purpose of studying the fluctuation, it is more convenient to gauge-transform Ψ_ε by $U_\varepsilon^{-1} [= (U_\varepsilon)^{-1} = (U^{-1})_\varepsilon]$ to consider

$$\mathcal{P}_\varepsilon = U_\varepsilon^{-1} (\Psi_\varepsilon + Q_B) U_\varepsilon = U_\varepsilon^{-1} (\Psi_\varepsilon - U_\varepsilon Q_B U_\varepsilon^{-1}) U_\varepsilon = \varepsilon \times \frac{1}{G_\varepsilon} c G_\varepsilon Bc (1 - G_\varepsilon). \quad (3.6)$$

Note that \mathcal{P}_ε is apparently of $O(\varepsilon)$ since, without the K_ε -regularization, the present gauge transformation transforms the pure-gauge UQ_BU^{-1} back to zero. The fluctuation around \mathcal{P}_ε and that around Ψ_ε are related by

$$\mathcal{S}_{\mathcal{P}_\varepsilon}[\Phi] = \mathcal{S}_{\Psi_\varepsilon}[U_\varepsilon \Phi U_\varepsilon^{-1}], \quad (3.7)$$

for \mathcal{S}_{Ψ_S} of (2.7). Note the following property of \mathcal{Q}_{Ψ_S} (2.8):

$$\mathcal{Q}_{V^{-1}(\Psi_S + Q_B)V}(V^{-1}AV) = V^{-1}(\mathcal{Q}_{\Psi_S}A)V. \quad (3.8)$$

The EOM of \mathcal{P}_ε is given by

$$Q_B \mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon^2 = U_\varepsilon^{-1} (Q_B \Psi_\varepsilon + \Psi_\varepsilon^2) U_\varepsilon = \varepsilon \times \frac{1}{G_\varepsilon} c (K_\varepsilon c G_\varepsilon - G_\varepsilon c K_\varepsilon) B c (1 - G_\varepsilon). \quad (3.9)$$

Though the EOM against the solution itself, (3.4), is not a gauge-invariant quantity, we have confirmed that $\int \mathcal{P}_\varepsilon * (Q_B \mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon^2)$ vanishes in the limit $\varepsilon \rightarrow 0$ for the two $G(K)$ in (3.5).

3.1 Six tachyon BV states around \mathcal{P}_ε

We are interested in whether physical fluctuations exist or not around the classical solutions \mathcal{P}_ε specified by two $G(K)$ in (3.5). Our expectation is of course that there are no physical fluctuations at all for G_{tv} , while there are quadruplicate of physical fluctuations for G_{2b} . In this paper, we consider this problem in the framework of the BV formalism by focusing on the tachyon mode. In the following, \mathcal{Q} denotes $\mathcal{Q}_{\mathcal{P}_\varepsilon}$, the BRST operator around \mathcal{P}_ε :

$$\mathcal{Q}A = \mathcal{Q}_{\mathcal{P}_\varepsilon} A = Q_B A + \mathcal{P}_\varepsilon * A - (-1)^A A * \mathcal{P}_\varepsilon. \quad (3.10)$$

Accordingly, $\mathcal{S}[\Phi]$ denotes $\mathcal{S}_{\mathcal{P}_\varepsilon}[\Phi]$, the action of the fluctuation Φ around \mathcal{P}_ε :

$$\mathcal{S}[\Phi] = \int \left(\frac{1}{2} \Phi * \mathcal{Q}\Phi + \frac{1}{3} \Phi^3 \right). \quad (3.11)$$

Our analysis proceeds as follows:

1. We first present a set of six BV states $\{u_i(k)\}$ containing the tachyon state. This set of BV states is similar to (2.24) for the photon longitudinal mode.
2. We evaluate the matrix $\omega_{ij}(k)$ (2.13) for the six BV states, and obtain the kinetic term of the action (3.11),

$$\mathcal{S}_0[\Phi] = \frac{1}{2} \int \Phi * \mathcal{Q}\Phi, \quad (3.12)$$

by expanding Φ in terms of the six states.

3. If the matrix ω_{ij} is non-degenerate, $\det \omega_{ij} \neq 0$, we conclude that the tachyon field is an *unphysical* one. On the other hand, if ω_{ij} is degenerate, $\det \omega_{ij} = 0$, and, furthermore, the kinetic term (3.12) is reduced to (1.2), the tachyon field is a *physical* one.

As a concrete choice of the six tachyon BV states $u_i(k)$, we take

$$\begin{aligned} N_{\text{gh}}=0: & \quad u_0 = L\hat{u}_0 R^{-1}, \\ N_{\text{gh}}=1: & \quad u_{1A} = L\hat{u}_{1A} R^{-1} + \xi [\mathcal{P}_\varepsilon, L\hat{u}_0 R^{-1}], \quad u_{1B} = L\hat{u}_{1B} R^{-1} - (1 - \xi) [\mathcal{P}_\varepsilon, L\hat{u}_0 R^{-1}], \\ N_{\text{gh}}=2: & \quad u_{2A} = L\hat{u}_{2A} R^{-1}, \quad u_{2B} = \{\mathcal{P}_\varepsilon, (1 - \xi) L\hat{u}_{1A} R^{-1} + \xi L\hat{u}_{1B} R^{-1}\}, \\ N_{\text{gh}}=3: & \quad u_3 = i [\mathcal{P}_\varepsilon, u_{2A}] = i [\mathcal{P}_\varepsilon, L\hat{u}_{2A} R^{-1}]. \end{aligned} \quad (3.13)$$

Each state in (3.13) consists of various ingredients. First, $\hat{u}_i(k)$ ($i = 0, 1A, 1B, 2A$) are defined by

$$\begin{aligned}\hat{u}_0 &= -i \frac{B}{K_\varepsilon} e^{-\alpha K_\varepsilon} c V_k e^{-\alpha K_\varepsilon}, \\ \hat{u}_{1A} &= e^{-\alpha K_\varepsilon} c V_k e^{-\alpha K_\varepsilon}, \\ \hat{u}_{1B} &= (1 - k^2) \frac{B}{K_\varepsilon} e^{-\alpha K_\varepsilon} c K c V_k e^{-\alpha K_\varepsilon} + \frac{\varepsilon}{K_\varepsilon} e^{-\alpha K_\varepsilon} c V_k e^{-\alpha K_\varepsilon}, \\ \hat{u}_{2A} &= e^{-\alpha K_\varepsilon} c K c V_k e^{-\alpha K_\varepsilon}.\end{aligned}\quad (3.14)$$

By Q_B , they are related by

$$i Q_B \hat{u}_0 = \hat{u}_{1A} - \hat{u}_{1B}, \quad Q_B \hat{u}_{1A} = Q_B \hat{u}_{1B} = (1 - k^2) \hat{u}_{2A}. \quad (3.15)$$

Note that there appear in (3.14) $e^{-\alpha K_\varepsilon}$ instead of $e^{-\alpha K}$. Namely, each state in (3.14) is multiplied by an extra factor $e^{-2\alpha\varepsilon}$. Though this is merely a c -number factor which is reduced to one in the limit $\varepsilon \rightarrow 0$, it makes the expressions of various $O(1/\varepsilon)$ quantities simpler as we will see in section 4.

Second, $L = L(K_\varepsilon)$ and $R = R(K_\varepsilon)$ in (3.13) are functions of only K_ε . Though they are quite arbitrary at this stage, we will determine later, for each classical solution \mathcal{P}_ε , their small K_ε behavior from the requirement that the EOM against $u_{1A/B}$ and that against the commutators $[u_0, u_{1A/B}]$ hold. Finally, ξ in (3.13) is a parameter related to the arbitrariness in the definitions of u_{1A} and u_{1B} .

The action of the BRST operator \mathcal{Q} (3.10) on the six states of (3.13) is given by

$$\begin{aligned}i \mathcal{Q} u_0 &= u_{1A} - u_{1B}, \\ \mathcal{Q} \begin{pmatrix} u_{1A} \\ u_{1B} \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} [(1 - k^2) u_{2A} + u_{2B}] + i \begin{pmatrix} \xi \\ -1 + \xi \end{pmatrix} [\text{EOM}_\varepsilon, u_0], \\ i \mathcal{Q} \begin{pmatrix} u_{2A} \\ u_{2B} \end{pmatrix} &= \begin{pmatrix} 1 \\ k^2 - 1 \end{pmatrix} u_3 + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} [\text{EOM}_\varepsilon, (1 - \xi) u_{1A} + \xi u_{1B}], \\ \mathcal{Q} u_3 &= i [\text{EOM}_\varepsilon, u_{2A}],\end{aligned}\quad (3.16)$$

where EOM_ε is defined by

$$\text{EOM}_\varepsilon = Q_B \mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon^2, \quad (3.17)$$

and given explicitly by (3.9).

The set of six BV states (3.13) has been constructed by comparing its BRST transformation property (3.16) with that of the six BV states (2.24) for the longitudinal photon and by taking into account that EOM_ε and \mathcal{P}_ε are both apparently of $O(\varepsilon)$. First, \hat{u}_{1A} is the tachyon state on the unstable vacuum, and \hat{u}_0 is \hat{u}_{1A} multiplied by the “homotopy operator” B/K_ε of Q_B . We start with u_0 , which is \hat{u}_0 dressed by L and R^{-1} , and divided $\mathcal{Q}u_0$ into the difference of u_{1A} and u_{1B} as given by the first equation of (3.16). If we ignore the apparently of $O(\varepsilon)$ terms, u_{1A} is the dressed tachyon state, and u_{1B} , which is multiplied by $(1 - k^2)$ vanishing at the tachyon on-shell $k^2 = 1$, corresponds to $k^2 u_{1B}$ in the first

equation of (2.28) for the massless longitudinal photon. We have distributed $[\mathcal{P}_\varepsilon, L\hat{u}_0 R^{-1}]$ in $i\mathcal{Q}u_0$ to u_{1A} and u_{1B} with coefficients specified by the parameter ξ . Then, we consider $\mathcal{Q}u_{1A}$ and $\mathcal{Q}u_{1B}$, which are equal to each other if $\mathcal{Q}^2 = 0$ and hence $\text{EOM}_\varepsilon = 0$ holds (see the second equation of (3.16)). We have chosen u_{2A} and u_{2B} as the part of $\mathcal{Q}u_{1A}$ which is multiplied by $(1 - k^2)$ and the rest, respectively. For u_{2A} and u_{2B} , we have no clear correspondence with the BV states of the longitudinal photon. Finally, u_3 is naturally defined from $\mathcal{Q}(u_{2A}, u_{2B})$ as given in the last equation of (3.16).

Our choice (3.13) of the six states u_i is of course not a unique one. For instance, in (3.14), the part $(\varepsilon/K_\varepsilon) e^{-\alpha K_\varepsilon} cV_k e^{-\alpha K_\varepsilon}$ in \hat{u}_{1B} may be moved to \hat{u}_{1A} to replace \hat{u}_{1A} , \hat{u}_{1B} and \hat{u}_{2A} in (3.13) with the following ones:

$$\begin{aligned}\hat{u}_{1A} &= \frac{K}{K_\varepsilon} e^{-\alpha K_\varepsilon} cV_k e^{-\alpha K_\varepsilon}, \\ \hat{u}_{1B} &= (1 - k^2) \frac{B}{K_\varepsilon} e^{-\alpha K_\varepsilon} cK cV_k e^{-\alpha K_\varepsilon}, \\ \hat{u}_{2A} &= \frac{K}{K_\varepsilon} e^{-\alpha K_\varepsilon} cK cV_k e^{-\alpha K_\varepsilon}.\end{aligned}\tag{3.18}$$

For u_{2A} and u_{2B} , we may take more generic linear combinations of the three terms $L\hat{u}_{2A}R^{-1}$, $\{\mathcal{P}_\varepsilon, L\hat{u}_{1A}R^{-1}\}$ and $\{\mathcal{P}_\varepsilon, L\hat{u}_{1B}R^{-1}\}$. However, here in this paper, we carry out the BV analysis by adopting the states of (3.13) with \hat{u}_i given by (3.14). In this sense, our analysis is rather an “experiment” and is not a comprehensive one. We do not know whether the conclusion of tachyon being physical or unphysical can be changed by taking another set of tachyon BV states.¹⁰

3.2 $\omega_{ij}^{(a,b)}(k)$

As we will see later, $L(K_\varepsilon)$ and $R^{-1}(K_\varepsilon)$ appearing in the definition of u_i (3.13) play a crucial role in making the EOM terms to vanish on the 2-brane. However, the pair (L, R) is not uniquely determined by this requirement alone. Therefore, we put a superscript (a) on (L, R) and the corresponding states u_i in (3.13) to distinguish different choices of (L, R) . For example, we write

$$u_0^{(a)} = L^{(a)} \hat{u}_0 (1/R^{(a)}).\tag{3.19}$$

Then, the matrix ω_{ij} (2.13) now has another index (a, b) :

$$\int u_i^{(a)}(k') u_j^{(b)}(k) = \omega_{ij}^{(a,b)}(k) \times (2\pi)^{26} \delta^{26}(k' + k),\tag{3.20}$$

with $i, j = 0, 1A, 1B, 2A, 2B, 3$. However, $\omega_{ij}^{(a,b)}(k)$ should not be regarded as a matrix with its left index (i, a) and right one (j, b) ; it is still a 6×6 matrix with a fixed pair of (a, b) . When we consider the action (3.11) in the final step of our analysis, we put $(a) = (b)$ by taking a particular (L, R) .

¹⁰In section 6, we argue the stability of the (un)physicalness of tachyon fluctuation under the change of the parameter ξ in (3.13) and under the replacement of \hat{u}_i (3.14) with those given by (3.18).

We see that all the components of $\omega_{ij}^{(a,b)}$ are not independent. By considering $\int u_0^{(a)} i \mathcal{Q} u_{2A}^{(b)}$, $\int u_0^{(a)} i \mathcal{Q} u_{2B}^{(b)}$ and $\int u_{1B}^{(a)} \mathcal{Q} u_{1A}^{(b)}$, and using

$$\int A_1 * \mathcal{Q} A_2 = -(-1)^{A_1} \int (\mathcal{Q} A_1) * A_2, \quad (3.21)$$

and (3.16), we obtain the following relations:

$$\omega_{1B,2A}^{(a,b)} = \omega_{1A,2A}^{(a,b)} + \omega_{0,3}^{(a,b)}, \quad (3.22)$$

$$\omega_{1B,2B}^{(a,b)} = \omega_{1A,2B}^{(a,b)} - (1 - k^2) \omega_{0,3}^{(a,b)}, \quad (3.23)$$

$$(1 - k^2) \omega_{1B,2A}^{(a,b)} + \omega_{1B,2B}^{(a,b)} = (1 - k^2) \omega_{1A,2A}^{(b,a)} + \omega_{1A,2B}^{(b,a)}. \quad (3.24)$$

In deriving the last two relations, we have assumed the vanishing of the EOM terms:

$$\int [u_0^{(a)}, u_i^{(b)}] * \text{EOM}_\varepsilon = 0, \quad (i = 1A, 1B). \quad (3.25)$$

From (3.22) and (3.23), we also have

$$(1 - k^2) \omega_{1B,2A}^{(a,b)} + \omega_{1B,2B}^{(a,b)} = (1 - k^2) \omega_{1A,2A}^{(a,b)} + \omega_{1A,2B}^{(a,b)}. \quad (3.26)$$

Therefore, among the five components, $\omega_{1A/B,2A/B}^{(a,b)}$ and $\omega_{0,3}^{(a,b)}$, we can choose $\omega_{1A,2A}^{(a,b)}$, $\omega_{1A,2B}^{(a,b)}$ and $\omega_{0,3}^{(a,b)}$ as independent ones, and write the submatrix $\Omega^{(a,b)}$ as

$$\Omega^{(a,b)} \equiv \begin{pmatrix} \omega_{1A,2A}^{(a,b)} & \omega_{1A,2B}^{(a,b)} \\ \omega_{1B,2A}^{(a,b)} & \omega_{1B,2B}^{(a,b)} \end{pmatrix} = \begin{pmatrix} \omega_{1A,2A}^{(a,b)} & \omega_{1A,2B}^{(a,b)} \\ \omega_{1A,2A}^{(a,b)} + \omega_{0,3}^{(a,b)} & \omega_{1A,2B}^{(a,b)} + (k^2 - 1) \omega_{0,3}^{(a,b)} \end{pmatrix}. \quad (3.27)$$

Its determinant is given by

$$|\Omega^{(a,b)}| = \omega_{0,3}^{(a,b)} [(k^2 - 1) \omega_{1A,2A}^{(a,b)} - \omega_{1A,2B}^{(a,b)}]. \quad (3.28)$$

Using (3.16) and assuming (3.25), we also obtain the following useful formulas:

$$\int u_i^{(a)} \mathcal{Q} u_j^{(b)} = (1 - k^2) \omega_{1A,2A}^{(a,b)} + \omega_{1A,2B}^{(a,b)}, \quad (i, j = 1A, 1B), \quad (3.29)$$

$$\int u_0^{(a)} i \mathcal{Q} (u_{2A}^{(b)}, u_{2B}^{(b)}) = (1, k^2 - 1) \omega_{0,3}^{(a,b)}, \quad (3.30)$$

where we have omitted $(2\pi)^{26} \delta^{26}(k' + k)$ on the r.h.s. .

3.3 Formulas for the EOM tests and $\omega_{ij}^{(a,b)}$

For the BV analysis for a given G_ε , we need to evaluate (i) the EOM test of \mathcal{P}_ε against $u_{1A/B}$ and $[u_0^{(a)}, u_{1A/B}^{(b)}]$, and (ii) $\omega_{0,3}^{(a,b)}$, $\omega_{1A,2A}^{(a,b)}$ and $\omega_{1A,2B}^{(a,b)}$. For u_{1A} and u_{1B} containing the parameter ξ (see (3.13)), it is convenient to introduce $w_\ell^{(a)}$ ($\ell = A, B, C$) defined by

$$w_A^{(a)} = L^{(a)} \hat{u}_{1A} R^{(a)-1}, \quad w_B^{(a)} = L^{(a)} \hat{u}_{1B} R^{(a)-1}, \quad w_C^{(a)} = [\mathcal{P}_\varepsilon, L^{(a)} \hat{u}_0 R^{(a)-1}], \quad (3.31)$$

and express $u_{1A/B}^{(a)}$ as

$$u_{1A}^{(a)} = w_A^{(a)} + \xi_a w_C^{(a)}, \quad u_{1B}^{(a)} = w_B^{(a)} - (1 - \xi_a) w_C^{(a)}, \quad (3.32)$$

where we have allowed the parameter ξ to depend on the index a of $L^{(a)}$ and $R^{(a)}$. For $L^{(a)}(K_\varepsilon)$ and $R^{(a)}(K_\varepsilon)$, we assume that their leading behaviors for $K_\varepsilon \sim 0$ are

$$L^{(a)}(K_\varepsilon) \sim K_\varepsilon^{m_a}, \quad R^{(a)}(K_\varepsilon) \sim K_\varepsilon^{n_a}, \quad (3.33)$$

and give their remaining K_ε -dependences as Laplace transforms:

$$L^{(a)}(K_\varepsilon) = K_\varepsilon^{m_a} \int_0^\infty ds_a v_L^{(a)}(s_a) e^{-K_\varepsilon s_a}, \quad \frac{1}{R^{(a)}(K_\varepsilon)} = \frac{1}{K_\varepsilon^{n_a}} \int_0^\infty d\tilde{s}_a v_{1/R}^{(a)}(\tilde{s}_a) e^{-K_\varepsilon \tilde{s}_a}. \quad (3.34)$$

As given in (3.34), we adopt s_a and \tilde{s}_a as the integration variable of the Laplace transform of $L^{(a)}$ and $1/R^{(a)}$, respectively. We adopt the following normalization for $v_L^{(a)}$ and $v_{1/R}^{(a)}$:

$$\int_0^\infty ds_a v_L^{(a)}(s_a) = \int_0^\infty d\tilde{s}_a v_{1/R}^{(a)}(\tilde{s}_a) = 1. \quad (3.35)$$

Namely, the coefficients of the leading terms (3.33) are taken to be equal to one. The pair (m_a, n_a) and the associated $v_L^{(a)}(s_a)$ and $v_{1/R}^{(a)}(\tilde{s}_a)$ should be determined by the requirement of the EOM as stated before. Concerning the choice of (m_a, n_a) , it would be natural to consider the case $m_a = n_a$ since the overall order of the BV states u_i (3.13) with respect to K_ε for $K_\varepsilon \sim 0$ is not changed from the case without $L^{(a)}$ and $R^{(a)}$. We will restrict ourselves to the case $m_a = n_a$ in the concrete calculations given in sections 4 and 5.

Then, the three kinds of quantities necessary for the BV analysis are expressed as the following integrations over the Laplace transform variables:

$$\int w_\ell * \text{EOM}_\varepsilon = \int_0^\infty ds v_L(s) \int_0^\infty d\tilde{s} v_{1/R}(\tilde{s}) E_\ell(s, \tilde{s}), \quad (3.36)$$

$$i \int [w_\ell^{(a)}(k'), u_0^{(b)}(k)] * \text{EOM}_\varepsilon = \int_{(s_a, s_b, \tilde{s}_a, \tilde{s}_b)} E_{\ell,0}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b) \times (2\pi)^{26} \delta^{26}(k' + k), \quad (3.37)$$

$$\omega_{ij}^{(a,b)}(k) = \int_{(s_a, s_b, \tilde{s}_a, \tilde{s}_b)} W_{ij}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b), \quad (3.38)$$

where $\int_{(s_a, s_b, \tilde{s}_a, \tilde{s}_b)}$ is the integration defined by

$$\int_{(s_a, s_b, \tilde{s}_a, \tilde{s}_b)} = \int_0^\infty ds_a v_L^{(a)}(s_a) \int_0^\infty ds_b v_L^{(b)}(s_b) \int_0^\infty d\tilde{s}_a v_{1/R}^{(a)}(\tilde{s}_a) \int_0^\infty d\tilde{s}_b v_{1/R}^{(b)}(\tilde{s}_b). \quad (3.39)$$

The explicit expressions of E_ℓ , $E_{\ell,0}^{(a,b)}$ ($\ell = A, B, C$) and $W_{ij}^{(a,b)}$ are lengthy and hence are summarized in appendix B. They are given as sliver frame integrations containing a single or no B . Though some of their defining expressions contain two or more B , we have used the KBc algebra to reduce them to sliver frame integrations with a single B .

The three E_ℓ are not independent, but they satisfy the following relation:

$$E_A - E_B + E_C = 0. \quad (3.40)$$

This follows from $w_A - w_B + w_C = u_{1A} - u_{1B} = i\mathcal{Q}u_0$ (see the first of (3.16)) and the Bianchi identity:

$$\mathcal{Q} \text{EOM}_\varepsilon = 0. \quad (3.41)$$

Eq. (3.40) can be used as a consistency check of the calculations.

3.4 The action of the fluctuation in the non-degenerate case

Let us consider the kinetic term $\mathcal{S}_0[\Phi]$ (3.12) in the case of non-degenerate ω_{ij} . We have attached the superscript (a, b) on $\omega_{ij}^{(a, b)}$ for distinguishing $(v_L, v_{1/R}, \xi)$ defining the state u_i and that defining u_j . However, when we express the fluctuation in terms of the basis $\{u_i\}$ and the corresponding component fields, we choose one particular $(v_L, v_{1/R}, \xi)$. Namely, when we consider the action (3.12), there appear only $\omega_{ij}^{(a, a)}$ with $(a) = (b)$. Therefore, we here omit the superscript (a, a) and simply write ω_{ij} .

When ω_{ij} is non-degenerate and the determinant (3.28) is not identically equal to zero, $|\Omega| \neq 0$, it is convenient to move to the Darboux basis by switching from (u_{2A}, u_{2B}) to (u_{2P}, u_{2Q}) defined by

$$(u_{2P}, u_{2Q}) = (u_{2A}, u_{2B}) \Omega^{-1}, \quad (3.42)$$

where the inverse matrix Ω^{-1} is given by

$$\Omega^{-1} = \frac{1}{|\Omega|} \begin{pmatrix} \omega_{1B, 2B} & -\omega_{1A, 2B} \\ -\omega_{1B, 2A} & \omega_{1A, 2A} \end{pmatrix}, \quad |\Omega| = \omega_{0, 3} [(k^2 - 1) \omega_{1A, 2A} - \omega_{1A, 2B}]. \quad (3.43)$$

The new set $\{u_0, u_{1A}, u_{1B}, u_{2P}, u_{2Q}, u_3\}$ is in fact a Darboux basis since we have

$$\begin{pmatrix} \omega_{1A, 2P} & \omega_{1A, 2Q} \\ \omega_{1B, 2P} & \omega_{1B, 2Q} \end{pmatrix} = \int \begin{pmatrix} u_{1A} \\ u_{1B} \end{pmatrix} (u_{2P}, u_{2Q}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.44)$$

Instead of (3.30), (u_{2P}, u_{2Q}) satisfies

$$\int u_0 i\mathcal{Q} (u_{2P}, u_{2Q}) = \omega_{0, 3} (1, k^2 - 1) \Omega^{-1} = (-1, 1). \quad (3.45)$$

For expressing Φ in terms of the Darboux basis, it is more convenient to use still another one $\{\tilde{u}_i\}$ with tilde, which is defined by multiplying the states corresponding to the fields and anti-fields by $\sqrt{\varpi(k)}$ and its inverse, respectively:

$$\begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_{1A} \\ \tilde{u}_{2Q} \end{pmatrix} = \sqrt{\varpi(k)} \begin{pmatrix} u_0 \\ u_{1A} \\ u_{2Q} \end{pmatrix}, \quad \begin{pmatrix} \tilde{u}_{1B} \\ \tilde{u}_{2P} \\ \tilde{u}_3 \end{pmatrix} = \frac{1}{\sqrt{\varpi(k)}} \begin{pmatrix} u_{1B} \\ u_{2P} \\ u_3 \end{pmatrix}, \quad (3.46)$$

with $\varpi(k)$ given by

$$\varpi(k) = (k^2 - 1) \omega_{1A, 2A}(k) - \omega_{1A, 2B}(k). \quad (3.47)$$

Then, expressing Φ as

$$\Phi = \int_k \left\{ \tilde{u}_0(k) C(k) + \tilde{u}_{1A}(k) \chi(k) + \tilde{u}_{1B}(k) \bar{C}_\star(k) + \tilde{u}_{2P}(k) \chi_\star(k) + \tilde{u}_{2Q}(k) \bar{C}(k) + \tilde{u}_3(k) C_\star(k) \right\}, \quad (3.48)$$

the kinetic term (3.12) is given in terms of the component fields and anti-fields as

$$\mathcal{S}_0[\Phi] = \int_k \left\{ -\frac{1}{2} (\varpi \chi + \bar{C}_\star)(-k) (\varpi \chi + \bar{C}_\star)(k) + i (\varpi \bar{C} - \chi_\star)(-k) C(k) \right\}. \quad (3.49)$$

The action (2.31) for the photon longitudinal mode on the unstable vacuum is essentially the special case of (3.49) with $\varpi(k) = k^2$, and the gauge-fixing process for (3.49) goes in the same manner as for (2.31). Adopting the gauge L with $\chi_\star = \bar{C}_\star = C_\star = 0$, the gauge-fixed action and the BRST transformation are given by

$$\hat{\mathcal{S}}_0 = \mathcal{S}_0|_L = \int_k \left\{ -\frac{1}{2} (\varpi \chi)(-k) (\varpi \chi)(k) + i (\varpi \bar{C})(-k) C(k) \right\}, \quad (3.50)$$

and

$$\begin{aligned} \hat{\delta}_B \chi(k) &= i \frac{\delta \mathcal{S}_0}{\delta \chi_\star(-k)} \Big|_L = C(k), \\ \hat{\delta}_B \bar{C}(k) &= i \frac{\delta \mathcal{S}_0}{\delta \bar{C}_\star(-k)} \Big|_L = -i \varpi(k) \chi(k), \\ \hat{\delta}_B C(k) &= i \omega_{0,3}(k)^{-1} \frac{\delta \mathcal{S}_0}{\delta C_\star(-k)} \Big|_L = 0. \end{aligned} \quad (3.51)$$

If $\varpi(k)$ has a zero at $k^2 = -m^2$, the action (3.50) describes a totally unphysical system with mass m explained in the Introduction.

The above argument leading to (3.49) does not apply if we ω_{ij} is degenerate. In such a case, the system can describe a physical one in general.

3.5 (Non-)hermiticity of the BV states

Our tachyon BV basis $\{u_i\}$ given by (3.13) has in fact a problem that it does not satisfy the hermiticity condition. We will explain it in this subsection.

In the original CSFT action (2.1), the the string field Ψ is assumed to be hermitian; $\Psi^\dagger = \Psi$, or more generally, $\Psi^\dagger = W(K) \Psi W(K)^{-1}$ with $W(K)$ depending only on K . This constraint ensures the reality of the action (2.1) and, at the same time, prevents the duplication of each fluctuation modes. Then, let us consider the hermiticity for the action (3.11) of the fluctuation Φ around \mathcal{P}_ε . First, \mathcal{P}_ε (3.6) satisfies the hermiticity in the following sense:

$$\mathcal{P}_\varepsilon^\dagger = \varepsilon \times (1 - G_\varepsilon) c G_\varepsilon B c \frac{1}{G_\varepsilon} = W \mathcal{P}_\varepsilon W^{-1}, \quad (3.52)$$

with W given by

$$W = G_\varepsilon (1 - G_\varepsilon). \quad (3.53)$$

Therefore, the fluctuation Φ in (3.11) must satisfy the same hermiticity:

$$\Phi^\dagger = W \Phi W^{-1}. \quad (3.54)$$

If (3.54) holds, it follows that $\mathcal{Q}\Phi$ with \mathcal{Q} defined by (3.10) also satisfies the same hermiticity¹¹

$$(\mathcal{Q}\Phi)^\dagger = W (\mathcal{Q}\Phi) W^{-1}, \quad (3.55)$$

and hence that the action (3.11) is real. In the expansion (2.11) of Φ in terms of the basis $\{u_i(k)\}$ and the component fields $\varphi^i(k)$, the hermiticity of Φ , (3.54), is realized by imposing

$$u_i(k)^\dagger = W u_i(-k) W^{-1}, \quad (3.56)$$

and $\varphi^i(k)^\dagger = \varphi^i(-k)$. However, our BV states (3.13) do not satisfy this hermiticity condition.

One way to realize the hermiticity (3.56) is to take, instead of the states u_i (3.13), the following ones U_i :

$$U_i(k) = \frac{1}{2} \left[u_i(k) + W^{-1} u_i(-k)^\dagger W \right]. \quad (3.57)$$

In fact, $U_i(k)$ satisfies (3.56) since W (3.53) is hermitian, $W^\dagger = W$. The relations (3.16) under the operation of \mathcal{Q} remain valid when u_i is replaced with U_i .

However, the results of the BV analysis which will be presented in sections 4 and 5 are largely changed if we adopt the hermitian basis $\{U_i\}$ instead of $\{u_i\}$. The EOM against $U_{1A/B}$ is the same as that for $u_{1A/B}$. On the other hand, the cross terms among the two terms on the r.h.s. of (3.57) are added to the EOM against the commutator $[U_{1A/B}^{(a)}, U_0^{(b)}]$ as well as $\omega_{ij}^{(a,b)}$ defined by (3.20) with u_i replaced with U_i . Sample calculations show that these cross terms change the results of sections 4 and 5 to much more complicated ones. For example, the EOM against $[U_{1A/B}^{(a)}, U_0^{(b)}]$ on the 2-brane no longer holds for any $(v_L, v_{1/R}, \xi)$ with $(m, n) = (1, 1)$. Therefore, we will continue our analysis by using the original non-hermitian basis $\{u_i\}$, though this is certainly a problem to be solved in the future.

4 BV analysis around the 2-brane solution

In this section, we carry out the BV analysis of the six states of (3.13) for the 2-brane solution given by G_{2b} in (3.5). Our analysis consists of the following three steps:

1. Evaluation of the EOM of \mathcal{P}_ε against $u_{1A/B}$ and $[u_0^{(a)}, u_{1A/B}^{(b)}]$ (recall (3.25) for the necessity of the latter). From the vanishing of these EOMs, we determine the allowed set of $(L(K_\varepsilon), R(K_\varepsilon))$.
2. Calculation of $\omega_{0,3}$, $\omega_{1A,2A}$ and $\omega_{1A,2B}$ for $(L(K_\varepsilon), R(K_\varepsilon))$ determined above. Our expectation is that $\omega_{0,3} = 0$, namely, that the present set of six BV states is degenerate and therefore the tachyon can be physical.
3. Derivation of the kinetic term $\mathcal{S}_0[\Phi]$ (3.12) of the fluctuation Φ in terms of the component fields defined by the basis $\{u_i\}$.

¹¹In deriving (3.55), we use the property $Q_B K = 0$ and $(Q_B A)^\dagger = -(-1)^A Q_B A^\dagger$ valid for any string field A .

4.1 EOM against u_{1A} and u_{1B}

First, let us consider the EOM test against u_{1A} and u_{1B} . For this purpose, we have to evaluate $E_\ell(s, \bar{s})$ of (3.36), which is given by (B.2)–(B.4) for a generic G_ε . For the 2-brane solution with $G = G_{2b}$ (3.5), E_ℓ are given explicitly by

$$E_A = -\varepsilon \int BcK_\varepsilon^{m-1} e^{-(\alpha+s)K_\varepsilon} c \frac{K_\varepsilon^{1-n}}{1+K_\varepsilon} e^{-(\alpha+\bar{s})K_\varepsilon} c \left[K_\varepsilon, 1 + \frac{1}{K_\varepsilon} \right]_c, \quad (4.1)$$

$$E_B = -\varepsilon \int BcKc \frac{K_\varepsilon^{1-n}}{1+K_\varepsilon} e^{-(\alpha+\bar{s})K_\varepsilon} c \left[K_\varepsilon, 1 + \frac{1}{K_\varepsilon} \right]_c \frac{e^{-(\alpha+s)K_\varepsilon}}{K_\varepsilon^{2-m}} - \varepsilon^2 \int Bc \frac{e^{-(\alpha+s)K_\varepsilon}}{K_\varepsilon^{2-m}} c \frac{K_\varepsilon^{1-n}}{1+K_\varepsilon} e^{-(\alpha+\bar{s})K_\varepsilon} c \left[K_\varepsilon, 1 + \frac{1}{K_\varepsilon} \right]_c, \quad (4.2)$$

$$E_C = \varepsilon^2 \int Bc \frac{K_\varepsilon^{1-n}}{1+K_\varepsilon} e^{-(\alpha+\bar{s})K_\varepsilon} c \left[K, \frac{1}{K_\varepsilon} \right]_c c \frac{e^{-(\alpha+s)K_\varepsilon}}{K_\varepsilon^{2-m}}. \quad (4.3)$$

where $[K_\varepsilon, 1 + (1/K_\varepsilon)]_c$ is defined by (see (B.1))

$$\left[K_\varepsilon, 1 + \frac{1}{K_\varepsilon} \right]_c = K_\varepsilon c \left(1 + \frac{1}{K_\varepsilon} \right) - \left(1 + \frac{1}{K_\varepsilon} \right) c K_\varepsilon. \quad (4.4)$$

Since the order of the correlator $\int BcK_\varepsilon^p cK_\varepsilon^q cK_\varepsilon^r cK_\varepsilon^s$ with respect to ε is $O(\varepsilon^{\min(p+q+r+s-3, 0)})$,¹² we find that

$$\begin{aligned} E_A &= O\left(\varepsilon^{1+\min(m-n-3+\delta_{m,1}+\delta_{n,1}+\delta_{m,1}\delta_{n,1}, 0)}\right), \\ E_B &= O\left(\varepsilon^{1+\min(m-n-3+2\delta_{n,1}+\delta_{n,0}, 0)}\right) + O\left(\varepsilon^{2+\min(m-n-4+\delta_{m,2}+\delta_{n,1}+\delta_{m,2}\delta_{n,1}, 0)}\right), \\ E_C &= O\left(\varepsilon^{2+\min(m-n-4+\delta_{n,1}, 0)}\right), \end{aligned} \quad (4.5)$$

where the Kronecker-delta terms are due to the identities $c^2 = cKcKc = 0$, and the two terms in E_B corresponds to those in (4.2). For a given (m, n) , E_ℓ can be evaluated by using the formulas of the $Bcccc$ correlators given in appendix A. However, we cannot carry out the calculation for a generic (m, n) , and the calculation for each (m, n) is very cumbersome. Therefore, we have evaluated E_ℓ only for two cases, $(m, n) = (1, 1)$ and $(0, 0)$. We have chosen $(m, n) = (1, 1)$ since, as seen from (4.5), E_ℓ are least singular with respect to ε for $(m, n) = (1, 1)$ if we restrict ourselves to the case $m = n$.¹³ We have taken the other one $(m, n) = (0, 0)$ including the simplest case $L = R = 1$ as a reference.

¹²This formula is derived by using the scaling property $G(\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4) = \lambda^3 G(t_1, t_2, t_3, t_4)$ of the correlator $G(t_1, t_2, t_3, t_4) = \int Bc e^{-t_1 K} c e^{-t_2 K} c e^{-t_3 K} c e^{-t_4 K}$ given by (A.4) and (A.6). For $p+q+r+s \geq 4$, the correlator contains divergences from $K = \infty$ and hence not regularized by ε . This is the reason why “min” appears in the formula.

¹³As we mentioned in section 3.3, the case $m = n$ is natural in the sense that the overall order of each BV state u_i (3.13) with respect to K_ε is not changed by (L, R) .

4.1.1 $(m, n) = (1, 1)$

In this case, E_ℓ are given up to $O(\varepsilon)$ terms by

$$\begin{aligned} E_A(s, \tilde{s}) &= 0, \\ E_B(s, \tilde{s}) &= E_C(s, \tilde{s}) = -\frac{3}{\pi^2} C_1(\tilde{s}) \times \frac{1}{\varepsilon} + \frac{1}{2} C_2(\tilde{s}), \end{aligned} \quad (4.6)$$

where $C_{1,2}(\tilde{s})$ are defined by

$$C_1(\tilde{s}) = \tilde{s} + \alpha + 1, \quad C_2(\tilde{s}) = (\tilde{s} + \alpha + 1)^2 + 1. \quad (4.7)$$

We defer further arguments on the EOM against $u_{1A/B}$ for $(m, n) = (1, 1)$ till we discuss the EOM against $[u_{1A/B}^{(a)}, u_0^{(b)}]$ in section 4.2.1.

4.1.2 $(m, n) = (0, 0)$

In this case, E_ℓ are all of $O(1/\varepsilon^2)$:

$$(E_A, E_B, E_C) = (1, 2, 1) \times \frac{3}{\pi^2} \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right). \quad (4.8)$$

This implies that we have to choose $\xi = -1$ to make the $1/\varepsilon^2$ part of the EOM test against u_{1A} and u_{1B} to vanish. Namely, we have to take

$$u_{1A} = w_A - w_C, \quad u_{1B} = w_B - 2w_C. \quad (4.9)$$

Then, the combinations of E_ℓ relevant to $u_{1A/B}$ are given as follows:

$$E_A - E_C = E_B - 2E_C = \left[\left(\frac{2}{\pi^2} + \frac{5}{3} \right) C_1(\tilde{s}) + \frac{3}{\pi^2} D_1(s) \right] \frac{1}{\varepsilon} - \frac{7}{2} C_2(\tilde{s}) + (2 - D_1(s)) C_1(\tilde{s}), \quad (4.10)$$

with $D_1(s)$ defined by

$$D_1(s) = s + \alpha. \quad (4.11)$$

4.2 EOM against $[u_{1A/B}^{(a)}, u_0^{(b)}]$

Next, we evaluate the EOM test against $[u_{1A/B}^{(a)}, u_0^{(b)}]$, namely, $E_{\ell,0}^{(a,b)}$ of (3.37) given explicitly by (B.5)–(B.7) for a generic G_ε . As in the previous subsection, we consider only the two cases; $(m_a, n_a) = (m_b, n_b) = (1, 1)$ and $(0, 0)$. We will explain the calculations in the case of $(1, 1)$ in great detail. The same method will be used also in the calculation of $\omega_{ij}^{(a,b)}$.

4.2.1 $(m_a, n_a) = (m_b, n_b) = (1, 1)$

For $G = G_{2b}$ (3.5) and for $(m_a, n_a) = (m_b, n_b) = (1, 1)$, $E_{A,0}^{(a,b)}$ (B.5) reads

$$\begin{aligned} &E_{A,0}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b) \\ &= \varepsilon \int Bc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{1+K_\varepsilon} c \left[K_\varepsilon, 1 + \frac{1}{K_\varepsilon} \right]_c \left[c, e^{-(\alpha+s_a)K_\varepsilon} \right] V_{-k} \frac{e^{-(2\alpha+\tilde{s}_a+s_b)K_\varepsilon}}{K_\varepsilon} \\ &\quad - \varepsilon \int Bc V_k e^{-(2\alpha+\tilde{s}_b+s_a)K_\varepsilon} c V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{1+K_\varepsilon} c \left[K_\varepsilon, 1 + \frac{1}{K_\varepsilon} \right]_c \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon}. \end{aligned} \quad (4.12)$$

Let us explain how we evaluate (4.12) and other $E_{\ell,0}^{(a,b)}$ for a generic momentum k_μ . Let us consider, as an example, the contribution of the $c e^{-(\alpha+s_a)K}$ term of the commutator $[c, e^{-(\alpha+s_a)K}]$ to the first integral of (4.12):

$$\begin{aligned} & \varepsilon \int Bc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{1+K_\varepsilon} c \left(Kc \frac{1}{K_\varepsilon} - \frac{1}{K_\varepsilon} cK \right) c e^{-(\alpha+s_a)K_\varepsilon} V_{-k} \frac{e^{-(2\alpha+\tilde{s}_a+s_b)K_\varepsilon}}{K_\varepsilon} \\ &= e^{-\varepsilon(\pi+\sum s)} \varepsilon \int_0^\infty dt_1 e^{-(1+\varepsilon)t_1} \int_0^\infty dt_2 e^{-\varepsilon t_2} \int_0^\infty dt_3 e^{-\varepsilon t_3} F(t_1, t_2, t_3), \end{aligned} \quad (4.13)$$

with $e^{-\varepsilon(\pi+\sum s)}$ defined by

$$e^{-\varepsilon(\pi+\sum s)} = e^{-\varepsilon(\pi+s_a+s_b+\tilde{s}_a+\tilde{s}_b)}. \quad (4.14)$$

In (4.13), t_1 , t_2 and t_3 are the Schwinger parameters for $1/(1+K_\varepsilon)$ and the two $1/K_\varepsilon$, respectively, and the function $F(t_1, t_2, t_3)$ is given by

$$\begin{aligned} F(t_1, t_2, t_3) &= - \frac{\partial}{\partial w_2} G(t_1 + \alpha + \tilde{s}_b, w_2, t_2, t_3 + 3\alpha + s_a + \tilde{s}_a + s_b; t_3 + 2\alpha + \tilde{s}_a + s_b) \Big|_{w_2=0} \\ &+ \frac{\partial}{\partial w_3} G(t_1 + \alpha + \tilde{s}_b, t_2, w_3, t_3 + 3\alpha + s_a + \tilde{s}_a + s_b; t_3 + 2\alpha + \tilde{s}_a + s_b) \Big|_{w_3=0}, \end{aligned} \quad (4.15)$$

where G is the product of the ghost correlator and the matter one on the infinite cylinder of circumference $\ell = w_1 + w_2 + w_3 + w_4$:

$$G(w_1, w_2, w_3, w_4; w_X) = \langle Bc(0)c(w_1)c(w_1+w_2)c(w_1+w_2+w_3) \rangle_\ell \times \left| \frac{\ell}{\pi} \sin \frac{\pi w_X}{\ell} \right|^{-2k^2}. \quad (4.16)$$

The explicit expressions of the correlators are given in appendix A.

One way to evaluate (4.13) in the limit $\varepsilon \rightarrow 0$ is to (i) make a change of integration variables from (t_2, t_3) for $1/K_\varepsilon$ to (u, x) by $(t_2, t_3) = (u/\varepsilon)(x, 1-x)$, (ii) carry out the x -integration first, (iii) Laurent-expand the integrand in powers of ε to a necessary order, and finally (iv) carry out the integrations over u and t_1 . In fact, we obtained the results (4.6), (4.8) and (4.10) by this method. However, it is hard to carry out explicitly the x -integration in (4.13) before Laurent-expanding with respect to ε due to the presence of the k^2 -dependent matter correlator in (4.16). On the other hand, Laurent-expanding the (t_1, u, x) -integrand with respect to ε before carrying out the x -integration sometimes leads to a wrong result. Namely, the integration regions where x or $1-x$ are of $O(\varepsilon)$ can make non-trivial contributions.

Our manipulation for obtaining the correct result for (4.13) is as follows. Eq. (4.13), which is multiplied by ε , can be non-vanishing due to negative powers of ε arising from the two $1/K_\varepsilon$ at the zero eigenvalue $K = 0$. In the r.h.s. of (4.13), this contribution comes from any of the following three regions of the (t_2, t_3) -integration:

$$\begin{aligned} \text{Region I: } & t_2 = \text{finite}, \quad t_3 \rightarrow \infty, \\ \text{Region II: } & t_2 \rightarrow \infty, \quad t_3 \rightarrow \infty, \\ \text{Region III: } & t_2 \rightarrow \infty, \quad t_3 = \text{finite}. \end{aligned} \quad (4.17)$$

Concretely, (4.13) is given as the sum of the contributions from the three regions:

$$e^{-\varepsilon(\pi+\sum s)} \int_0^\infty dt_1 e^{-(1+\varepsilon)t_1} [(I) + (II) + (III)], \quad (4.18)$$

with each term given by

$$(I) = \int_\varepsilon^\infty du e^{-u} \int_0^{\zeta u/\varepsilon} dy \operatorname{Ser}_\varepsilon F(t_1, t_2 = y, t_3 = (u/\varepsilon) - y), \quad (4.19)$$

$$(II) = \int_\varepsilon^\infty du e^{-u} \int_\zeta^{1-\eta} dx \operatorname{Ser}_\varepsilon \frac{u}{\varepsilon} F(t_1, t_2 = xu/\varepsilon, t_3 = (1-x)u/\varepsilon), \quad (4.20)$$

$$(III) = \int_\varepsilon^\infty du e^{-u} \int_0^{\eta u/\varepsilon} dy \operatorname{Ser}_\varepsilon F(t_1, t_2 = (u/\varepsilon) - y, t_3 = y), \quad (4.21)$$

where $\operatorname{Ser}_\varepsilon$ denotes the operation of Laurent-expanding the function with respect to ε to a necessary order.¹⁴ In each region, we have put $t_2 + t_3 = u/\varepsilon$ and limited the integration region of u to (ε, ∞) since the other region $(0, \varepsilon)$ cannot develop a negative power of ε . As given in (4.19)–(4.21), the three regions of (4.17) are specified by two parameters, ζ and η , which we assume to be of $O(\varepsilon^0)$. Explicitly, the evaluation of the terms (I)–(III) goes as follows:

Term (I). For (4.19), the Laurent expansion gives

$$\operatorname{Ser}_\varepsilon F = \frac{2\pi^2}{3} \left(\frac{\varepsilon}{u}\right)^3 (t_1 + \tilde{s}_b + \alpha) (t_1 + \tilde{s}_b + \alpha + y) (t_1 + s_a + \tilde{s}_b + 2\alpha + y)^{-2k^2} y^3 + \dots \quad (4.22)$$

The leading term of the y -integration is of order $(u/\varepsilon)^{\max(5-2k^2, 0)}$, and we obtain

$$(I) \sim \int_\varepsilon^\infty du e^{-u} \left(\frac{\varepsilon}{u}\right)^{3-\max(5-2k^2, 0)} = O(\varepsilon^{\min(2k^2-2, 1)}), \quad (4.23)$$

where we have used that¹⁵

$$\int_\varepsilon^\infty du e^{-u} \left(\frac{\varepsilon}{u}\right)^g = O(\varepsilon^{\min(g, 1)}). \quad (4.24)$$

The subleading term of (4.22), which is of $O((\varepsilon/u)^4)$, gives terms of order $\varepsilon^{\min(2k^2-2+p, 1)}$ with $p = 1, 2, \dots$.

Term (II). The Laurent expansion in (4.20) gives

$$\operatorname{Ser}_\varepsilon \frac{u}{\varepsilon} F = 2(t_1 + \tilde{s}_b + \alpha) \frac{\pi x \cos \pi x - \sin \pi x}{\sin \pi x} \left(\frac{u \sin \pi x}{\pi \varepsilon}\right)^{2-2k^2} + \dots \quad (4.25)$$

Since the x -integration in the range $\zeta \leq x \leq 1 - \eta$ is finite, we obtain

$$(II) \sim \int_\varepsilon^\infty du e^{-u} \left(\frac{u}{\varepsilon}\right)^{2-2k^2} = O(\varepsilon^{\min(2k^2-2, 1)}). \quad (4.26)$$

¹⁴In this Laurent expansion, we treat u , y , x and $1 - x$ as quantities of $O(\varepsilon^0)$.

¹⁵Precisely, the r.h.s. of (4.24) for $g = 1$ should read $O(\varepsilon \ln \varepsilon)$.

Term (III). The Laurent expansion in (4.21) gives

$$\text{Ser } F = -2(\tilde{s}_b + t_1 + \alpha)(s_b + \tilde{s}_a + 2\alpha + y)^{-2k^2}(s_a + s_b + \tilde{s}_a + 3\alpha + y) + \dots \quad (4.27)$$

Carrying out the y -integration, we get

$$\begin{aligned} \text{(III)} &= \frac{t_1 + \tilde{s}_b + \alpha}{(k^2 - 1)(2k^2 - 1)} \int_{\varepsilon}^{\infty} du e^{-u} \left[(s_b + \tilde{s}_a + 2\alpha + y)^{1-2k^2} \right. \\ &\quad \times \left. \left\{ 2(k^2 - 1)(s_a + \alpha) + (2k^2 - 1)(s_b + \tilde{s}_a + 2\alpha + y) \right\} \right]_{y=0}^{y=\eta u/\varepsilon} \\ &= -\frac{(t_1 + \tilde{s}_b + \alpha)(s_b + \tilde{s}_a + 2\alpha)^{1-2k^2}}{(k^2 - 1)(2k^2 - 1)} \left\{ 2(k^2 - 1)(s_a + \alpha) + (2k^2 - 1)(s_b + \tilde{s}_a + 2\alpha) \right\} \\ &\quad + O(\varepsilon^{\min(2k^2-2,1)}), \end{aligned} \quad (4.28)$$

where the last term is the contribution of the $y = \eta u/\varepsilon$ term.

Summing the three terms, (4.23), (4.26) and (4.28), and carrying out the t_1 -integration of (4.18), we finally find that (4.13) is given by

$$\begin{aligned} &-\frac{(\tilde{s}_b + \alpha + 1)(s_b + \tilde{s}_a + 2\alpha)^{1-2k^2}}{(k^2 - 1)(2k^2 - 1)} \left\{ 2(k^2 - 1)(s_a + \alpha) + (2k^2 - 1)(s_b + \tilde{s}_a + 2\alpha) \right\} \\ &+ O(\varepsilon^{\min(2k^2-2,1)}). \end{aligned} \quad (4.29)$$

This result can also be checked by numerically carrying out the integrations of (4.13) for given values of ε , k^2 and other parameters in (4.13).

The evaluation of the other term of the first integral of (4.12), namely, the term containing the $e^{-(\alpha+s_a)K}c$ part of the commutator $[c, e^{-(\alpha+s_a)}]$, is quite similar. In fact, the two terms of the commutator almost cancel one another, and the whole of the first integral of (4.12) turns out to be simply of $O(\varepsilon^{\min(2k^2-1,1)})$.¹⁶

Next, the second integral of (4.12) is given by

$$\begin{aligned} &e^{-\varepsilon(\pi+\Sigma s)} \varepsilon \int_0^{\infty} dt_1 e^{-(1+\varepsilon)t_1} \int_0^{\infty} dt_2 e^{-\varepsilon t_2} \left\{ G(2\alpha + \tilde{s}_b + s_a, t_1 + \alpha + \tilde{s}_a, t_2, \alpha + s_b; 2\alpha + \tilde{s}_b + s_a) \right. \\ &\quad \left. + (1 + t_3) \partial_{w_3} G(2\alpha + \tilde{s}_b + s_a, t_1 + \alpha + \tilde{s}_a, w_3, t_2 + \alpha + s_b; 2\alpha + \tilde{s}_b + s_a) \right|_{w_3=0} \}. \end{aligned} \quad (4.30)$$

The evaluation of this term is much easier than that of the first integral explained above since there is only one Schwinger parameter t_2 for $1/K_{\varepsilon}$. We have only to Laurent-expand the integrand with respect to ε after making the change of integration variables from t_2 to $u = \varepsilon t_2$, and carry out the (t_1, u) -integrations. After all, the whole of $E_{A,0}^{(a,b)}$ (4.12) is found to be given by

$$E_{A,0}^{(a,b)} = O(\varepsilon^{\min(2k^2-1,1)}) + (s_a + \tilde{s}_b + 2\alpha)^{1-2k^2} \left[C_2(\tilde{s}_a) + (s_a + \tilde{s}_b + 2\alpha) C_1(\tilde{s}_a) \right], \quad (4.31)$$

where the first (second) term on the r.h.s. corresponds to the first (second) integral of (4.12).

¹⁶The actual ε -dependence may be a milder one since we are not taking into account the possibility of cancellations among the three terms (4.19)–(4.21) for the whole of the first integral of (4.12). In fact, numerical analysis supports a milder behavior $O(\varepsilon^{\min(2k^2,1)})$.

The first term on the r.h.s. of (4.31), $O(\varepsilon^{\min(2k^2-1,1)})$, vanishes in the limit $\varepsilon \rightarrow 0$ for $k^2 > 1/2$, while it is divergent for $k^2 < 1/2$. Here, we define $E_{A,0}^{(a,b)}$ for a generic k^2 as the “analytic continuation” from the region of sufficiently large k^2 ($k^2 > 1/2$ in the present case). Thus, $E_{A,0}^{(a,b)}$ is simply given by the last term of (4.31). Eq. (4.31) has been obtained by keeping only the first term of the Laurent expansion. The subleading term which has an extra positive power $(\varepsilon/u)^p$ contributes $O(\varepsilon^{\min(2k^2-1+p,1)})$. This vanishes for $k^2 > 1/2$ and does not affect our definition of $E_{A,0}^{(a,b)}$ by analytic continuation. We apply this definition of $E_{A,0}^{(a,b)}$ by analytic continuation from the region of sufficiently large k^2 also to other k^2 -dependent quantities; $E_{\ell,0}^{(a,b)}$ ($\ell = B, C$) and $W_{ij}^{(a,b)}$.

The evaluation of $E_{B,0}^{(a,b)}$ and $E_{C,0}^{(a,b)}$ is similar except two points. First, they contain terms with *three* $1/K_\varepsilon$. For such terms, we have to carry out the integration over the three Schwinger parameters by considering $2^3 - 1 = 7$ regions with at least one large parameters (see appendix C). Second, the obtained $E_{\ell,0}^{(a,b)}$ ($\ell = B, C$) both contain $1/\varepsilon$ terms, and, therefore, $e^{-\varepsilon(\pi+\Sigma s)}$ (4.14) multiplying them makes non-trivial contribution to their $O(\varepsilon^0)$ terms. Then, we get the following results:¹⁷

$$E_{B,0}^{(a,b)} = \frac{1-k^2}{1-2k^2} (s_a + \tilde{s}_b + 2\alpha)^{1-2k^2} C_2(\tilde{s}_a) + (s_b + \tilde{s}_a + 2\alpha)^{1-2k^2} \left\{ -\frac{1}{2} C_2(\tilde{s}_b) + \left[\frac{1}{1-2k^2} \frac{3}{\pi^2} \frac{1}{\varepsilon} - (s_b + \tilde{s}_a + 2\alpha) \right] C_1(\tilde{s}_b) \right\}, \quad (4.32)$$

$$E_{C,0}^{(a,b)} = -\frac{1}{2(1-2k^2)} (s_b + \tilde{s}_a + 2\alpha)^{1-2k^2} [2 C_1(\tilde{s}_b) + C_2(\tilde{s}_b)] - \frac{1}{1-2k^2} \left(\frac{3}{\pi^2} \frac{1}{\varepsilon} + 1 \right) (s_a + \tilde{s}_b + 2\alpha)^{1-2k^2} C_1(\tilde{s}_a). \quad (4.33)$$

In particular, $E_{A,0}^{(a,b)} - E_{B,0}^{(a,b)} + E_{C,0}^{(a,b)}$, which is related to the EOM against $[u_0^{(a)}, u_{1A}^{(b)} - u_{1B}^{(b)}]$, is given by

$$E_{A,0}^{(a,b)} - E_{B,0}^{(a,b)} + E_{C,0}^{(a,b)} = - \left\{ \frac{k^2}{1-2k^2} C_2(\tilde{s}_a) + \left[\frac{1}{1-2k^2} \left(\frac{3}{\pi^2} \frac{1}{\varepsilon} + 1 \right) - S_{a+\tilde{b}} \right] C_1(\tilde{s}_a) \right\} S_{a+\tilde{b}}^{1-2k^2} - \left\{ \frac{k^2}{1-2k^2} C_2(\tilde{s}_b) + \left[\frac{1}{1-2k^2} \left(\frac{3}{\pi^2} \frac{1}{\varepsilon} + 1 \right) - S_{b+\tilde{a}} \right] C_1(\tilde{s}_b) \right\} S_{b+\tilde{a}}^{1-2k^2}, \quad (4.34)$$

where $S_{a+\tilde{b}}$ and $S_{b+\tilde{a}}$ are defined by

$$S_{a+\tilde{b}} = s_a + \tilde{s}_b + 2\alpha, \quad S_{b+\tilde{a}} = s_b + \tilde{s}_a + 2\alpha. \quad (4.35)$$

Eq. (4.34) implies that, in order for the EOM against $[u_0^{(a)}, u_{1A}^{(b)} - u_{1B}^{(b)}]$ to hold for any k^2 , $v_{1/R}^{(a)}(\tilde{s}_a)$ and $v_{1/R}^{(b)}(\tilde{s}_b)$ must be such that satisfy

$$\int_0^\infty d\tilde{s} v_{1/R}(\tilde{s}) \begin{pmatrix} C_1(\tilde{s}) \\ C_2(\tilde{s}) \end{pmatrix} = 0. \quad (4.36)$$

¹⁷If we adopt $e^{-\alpha K}$ instead of $e^{-\alpha K_\varepsilon}$ in the definition of \hat{u}_i (3.14), we have to replace all $1/\varepsilon$ in (4.32) and (4.33) with $(1/\varepsilon) + \pi$.

In this case, the EOMs against $[u_0^{(a)}, u_{1A}^{(b)}]$ and $[u_0^{(a)}, u_{1B}^{(b)}]$ hold for any ξ . Furthermore, the EOM against $u_{1A/B}$ also holds for any ξ as seen from (4.6).

The condition (4.36) restricts the first few terms of the series expansion of $1/R(K_\varepsilon)$ with respect to K_ε . In fact, expanding the expression (3.34) for $1/R(K_\varepsilon)$ in powers of K_ε and using the condition (4.36), we obtain

$$\frac{1}{R(K_\varepsilon)} = \frac{1}{K_\varepsilon} \left\{ 1 + (\alpha + 1)K_\varepsilon + \frac{1}{2}\alpha(\alpha + 2)K_\varepsilon^2 + O(K_\varepsilon^3) \right\}. \quad (4.37)$$

4.2.2 $(m_a, n_a) = (m_b, n_b) = (0, 0)$

The complete evaluation of $E_{\ell,0}^{(a,b)}$ for $(m_a, n_a) = (m_b, n_b) = (0, 0)$ is much harder than that for $(1, 1)$. Here, however, we need only their $1/\varepsilon^2$ part:

$$\begin{aligned} \begin{pmatrix} E_{A,0}^{(a,b)} \\ E_{B,0}^{(a,b)} \end{pmatrix} &= - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{3}{\pi^2} \frac{1}{1 - 2k^2} (s_b + \tilde{s}_a + 2\alpha)^{1-2k^2} \times \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right), \\ E_{C,0}^{(a,b)} &= \frac{3}{\pi^2} \frac{1}{1 - 2k^2} (s_a + \tilde{s}_b + 2\alpha)^{1-2k^2} \times \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right). \end{aligned} \quad (4.38)$$

This result implies that the $1/\varepsilon^2$ part of the EOM against $[u_{1A/B}^{(a)}, u_0^{(b)}]$ cannot vanish for any choice of ξ_a (and, in particular, for $\xi_a = -1$ determined in section 4.1.2) at least in the case $(a) = (b)$ which we take in the end. Therefore, we do not consider the case $(m_a, n_a) = (m_b, n_b) = (0, 0)$ in the rest of this section.

4.3 $\omega_{ij}^{(a,b)}$ for $(m_a, n_a) = (m_b, n_b) = (1, 1)$

Let us complete the BV analysis around the 2-brane solution by evaluating the matrix $\omega_{ij}^{(a,b)}$ (3.20) for $(m_a, n_a) = (m_b, n_b) = (1, 1)$ (see (B.8)–(B.16) for explicit expressions of $W_{i,j}^{(a,b)}$). First, for $W_{0,3}^{(a,b)}$ (B.8), we obtain¹⁸

$$W_{0,3}^{(a,b)} = O(\varepsilon^{\min(2k^2, 1)}). \quad (4.39)$$

Namely, $W_{0,3}^{(a,b)}$ defined by analytic continuation is identically equal to zero. Next, $W_{1A,2A}^{(a,b)(1)}$ (B.10) and $W_{1A,2A}^{(a,b)(2)}$ (B.11) constituting $W_{1A,2A}^{(a,b)}$ by (B.9) are given by

$$W_{1A,2A}^{(a,b)(1)} = \int c V_{-k} e^{-(2\alpha + s_b + \tilde{s}_a)K} c K c V_k e^{-(2\alpha + s_a + \tilde{s}_b)K} = f(s_a, s_b, \tilde{s}_a, \tilde{s}_b), \quad (4.40)$$

$$W_{1A,2A}^{(a,b)(2)} = O(\varepsilon^{\min(2k^2, 1)}), \quad (4.41)$$

with

$$f(s_a, s_b, \tilde{s}_a, \tilde{s}_b) = \left[\left(1 + \frac{s_a + s_b + \tilde{s}_a + \tilde{s}_b}{\pi} \right) \sin \left(\frac{\frac{\pi}{2} + s_b + \tilde{s}_a}{1 + \frac{1}{\pi}(s_a + s_b + \tilde{s}_a + \tilde{s}_b)} \right) \right]^{2(1-k^2)}. \quad (4.42)$$

¹⁸The first and the second terms in (B.8) are of $O(\varepsilon^{\min(2k^2, 1)})$ and $O(\varepsilon)$, respectively.

Therefore, $\omega_{1A,2A}^{(a,b)}$ defined by analytic continuation is independent of ξ_a and is given by (see (3.38))

$$\omega_{1A,2A}^{(a,b)} = \int_{(s_a, s_b, \tilde{s}_a, \tilde{s}_b)} f(s_a, s_b, \tilde{s}_a, \tilde{s}_b). \quad (4.43)$$

Finally, for $W_{1A,2B}^{(a,b)}$ given by (B.12), we obtain the following result after the analytic continuation:

$$\begin{aligned} W_{1A,2B}^{(a,b)(1)} &= 0, \\ W_{1A,2B}^{(a,b)(2)} &= \frac{1}{2} S_{a+\tilde{b}}^{1-2k^2} [C_2(\tilde{s}_a) + S_{a+\tilde{b}} C_1(\tilde{s}_a)], \\ W_{1A,2B}^{(a,b)(3)} &= -\frac{1}{2} S_{b+\tilde{a}}^{1-2k^2} [C_2(\tilde{s}_b) + S_{b+\tilde{a}} C_1(\tilde{s}_b)], \\ W_{1A,2B}^{(a,b)(4)} &= \frac{1}{2} \frac{1-k^2}{1-2k^2} \left[S_{a+\tilde{b}}^{1-2k^2} C_2(\tilde{s}_a) - S_{b+\tilde{a}}^{1-2k^2} C_2(\tilde{s}_b) \right] - \left(\frac{1}{1-2k^2} \frac{3}{\pi^2} \frac{1}{\varepsilon} - S_{a+\tilde{b}} \right) S_{a+\tilde{b}}^{1-2k^2} C_1(\tilde{s}_a). \end{aligned} \quad (4.44)$$

Assuming that $v_{1/R}^{(a)}$ and $v_{1/R}^{(b)}$ both satisfy the condition (4.36), our result (4.44) implies that $\omega_{1A,2B}^{(a,b)}$ is equal to zero for any (ξ_a, ξ_b) .

Summarizing, we have obtained

$$\omega_{0,3}^{(a,b)} = 0, \quad \omega_{1A,2A}^{(a,b)} = 1 + O(k^2 - 1), \quad \omega_{1A,2B}^{(a,b)} = 0, \quad (4.45)$$

and, from (3.27),

$$\begin{pmatrix} \omega_{1A,2A}^{(a,b)} & \omega_{1A,2B}^{(a,b)} \\ \omega_{1B,2A}^{(a,b)} & \omega_{1B,2B}^{(a,b)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \omega_{1A,2A}^{(a,b)}. \quad (4.46)$$

4.4 The action of the fluctuation with $(m, n) = (1, 1)$

The above result, in particular, $\omega_{0,3}^{(a,b)} = 0$, implies that the present $\omega_{ij}^{(a,b)}$ for the six BV-states is degenerate. From $\omega_{0,3}^{(a,b)} = 0$ and (4.46), we see that the rank of the 6×6 matrix $\omega_{ij}^{(a,b)}$ is two, and that there exists effectively the following four equivalences:¹⁹

$$u_0^{(a)} \sim 0, \quad u_{2B}^{(a)} \sim 0, \quad u_3^{(a)} \sim 0, \quad u_{1A}^{(a)} \sim u_{1B}^{(a)}. \quad (4.47)$$

Therefore, we express the fluctuation Φ around the solution \mathcal{P}_ε in terms of only u_{1A} and u_{2A} which are non-trivial and independent:

$$\Phi = \int_k (u_{1A}(k) \chi(k) + u_{2A}(k) \chi_\star(k)). \quad (4.48)$$

Here, we have chosen as $v_{1/R}(\tilde{s})$ defining u_i a suitable one satisfying the condition (4.36), and omitted the superscript (a) as in section 3.4. Plugging (4.48) into the kinetic term $\mathcal{S}_0[\Phi]$ (3.12) and using (3.29), we obtain

$$\mathcal{S}_0[\Phi] = - \int_k \frac{1}{2} \omega_{1A,2A}(k) (k^2 - 1) \chi(-k) \chi(k). \quad (4.49)$$

¹⁹For a state w , $w \sim 0$ implies that $\omega_{w,j}^{(a,b)} = \int w^{(a)} u_j^{(b)}$ vanishes for any $u_j^{(b)}$ in the six BV states. Note that $\mathcal{Q}w \sim 0$ follows from $w \sim 0$ due to the property (3.21).

Since we have $\omega_{1A,2A}(k^2 = 1) = 1$, the expansion (4.48) and the action (4.49) are essentially the same as (2.37) and (2.38), respectively, for the tachyon field on the unstable vacuum. The present χ represents a physical tachyon field.

Finally, let us interpret the above result in the context of the BRST cohomology problem. Using the truncation (4.47) and discarding the EOM terms in the BRST transformation formula (3.16), we obtain the following equations for the remaining u_{1A} and u_{2A} :

$$\mathcal{Q} u_{1A} = (1 - k^2) u_{2A}, \quad i\mathcal{Q} u_{2A} = 0. \quad (4.50)$$

The first equation and the fact that $u_0 \sim 0$, namely, that there is no candidate BRST parent of u_{1A} , imply that u_{1A} at $k^2 = 1$ is a physical state belonging to $\text{Ker } \mathcal{Q} / \text{Im } \mathcal{Q}$.

5 BV analysis around the tachyon vacuum solution

In this section, we repeat the BV analysis of the previous section by taking G_{tv} (3.5) which represents the tachyon vacuum. We expect of course that the matrix ω_{ij} of the six BV states u_i is non-degenerate and therefore the excitations they describe are unphysical ones. As (m, n) for (L, R) , we consider here again only the two cases, $(1, 1)$ and $(0, 0)$.

5.1 EOM against $u_{1A/B}$ and $[u_{1A/B}^{(a)}, u_0^{(b)}]$

First, E_ℓ ($\ell = A, B, C$) (3.36) for the EOM against $u_{1A/B}$ are calculated to be given by

$$E_A = E_B = 2, \quad E_C = 0 \quad \text{for} \quad (m, n) = (1, 1), \quad (5.1)$$

and

$$E_A = E_B = E_C = 0 \quad \text{for} \quad (m, n) = (0, 0). \quad (5.2)$$

The result (5.1) implies that the EOMs against u_{1A} and u_{1B} cannot be satisfied for any ξ in the case $(m, n) = (1, 1)$. On the other hand, EOMs against $u_{1A/B}$ both hold for an arbitrary ξ in the case $(0, 0)$. Therefore, in the rest of this section, we consider only the latter case $(m, n) = (0, 0)$.

Next, $E_{\ell,0}^{(a,b)}$ ($\ell = A, B, C$) (3.37) for the EOM against $[u_{1A/B}^{(a)}, u_0^{(b)}]$ in the case $(m, n) = (0, 0)$ are found to be given by

$$E_{A,0}^{(a,b)} = O(\varepsilon^{\min(2k^2, 1)}), \quad E_{B,0}^{(a,b)} = O(\varepsilon^{\min(2k^2-1, 1)}), \quad E_{C,0}^{(a,b)} = O(\varepsilon^{\min(2k^2-1, 1)}). \quad (5.3)$$

Namely, $E_{\ell,0}^{(a,b)}$ defined by analytic continuation are all equal to zero.

Summarizing, all the EOM tests are satisfied for any $(v_L, v_{1/R}, \xi)$ in the case $(m, n) = (0, 0)$. The choice $(m, n) = (1, 1)$ is excluded by the EOM test against $u_{1A/B}$.

5.2 $\omega_{ij}^{(a,b)}$ for $(m_a, n_a) = (m_b, n_b) = (0, 0)$ and $\xi_a = \xi_b = 0$

For the tachyon vacuum solution and for $(m_a, n_a) = (m_b, n_b) = (0, 0)$, we find that $W_{0,3}^{(a,b)}$ (B.8) is non-trivial and is given by

$$W_{0,3}^{(a,b)} = -(s_b + \tilde{s}_a + 2\alpha)^{2(1-k^2)} + O(\varepsilon^{\min(2k^2, 1)}), \quad (5.4)$$

where the last term should be discarded by analytic continuation.

Next, for $\omega_{1A,2A}^{(a,b)}$ and $\omega_{1A,2B}^{(a,b)}$, we have to specify (ξ_a, ξ_b) . Here, we consider the simplest case of $(\xi_a, \xi_b) = (0, 0)$, for which we need to calculate only $W_{1A,2A}^{(a,b)(1)}$ (B.10) and $W_{1A,2B}^{(a,b)(1)}$ (B.13). We see that the former, which is independent of G_ε and depends on (m_a, n_a) and (m_b, n_b) only through the differences $m_b - n_a$ and $m_a - n_b$, is the same as (4.40) for the 2-brane solution:

$$W_{1A,2A}^{(a,b)(1)} = f(s_a, s_b, \tilde{s}_a, \tilde{s}_b), \quad (5.5)$$

with f given by (4.42). As for the latter, we find that

$$W_{1A,2B}^{(a,b)(1)} = O(\varepsilon). \quad (5.6)$$

Our result implies that

$$\omega_{0,3}^{(a,b)} = -1 + O(k^2 - 1), \quad \omega_{1A,2A}^{(a,b)} = 1 + O(k^2 - 1), \quad \omega_{1A,2B}^{(a,b)} = 0, \quad (5.7)$$

for any $v_L^{(a/b)}$ and $v_{1/R}^{(a/b)}$. Since ω_{ij} for the six BV states u_i are non-degenerate, the general argument of section 3.4 does apply to the present case. From (5.7), the function $\varpi(k)$ (3.47) is given by²⁰

$$\varpi(k) = (k^2 - 1) \omega_{1A,2A}(k) = k^2 - 1 + O((k^2 - 1)^2), \quad (5.8)$$

and the fluctuation around the tachyon vacuum we have constructed is an unphysical one with $m^2 = -1$.

Finally, our result is interpreted in the BRST cohomology problem as follows. On the mass-shell $k^2 = 1$, ω_{ij} is reduced to

$$\omega_{0,3} = -1, \quad \begin{pmatrix} \omega_{1A,2A} & \omega_{1A,2B} \\ \omega_{1B,2A} & \omega_{1B,2B} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (k^2 = 1). \quad (5.9)$$

From this we find that $u_{1B} \sim 0$ and $u_{2B} \sim 0$ at $k^2 = 1$. Then, from (3.16), we obtain the following BRST transformation rule for the remaining $(u_0, u_{1A}, u_{2A}, u_3)$:

$$i\mathcal{Q}u_0 = u_{1A}, \quad \mathcal{Q}u_{1A} = 0, \quad i\mathcal{Q}u_{2A} = u_3, \quad \mathcal{Q}u_3 = 0, \quad (k^2 = 1). \quad (5.10)$$

This implies, in particular, that the candidate physical state u_{1A} is a trivial element of $\text{Ker } \mathcal{Q} / \text{Im } \mathcal{Q}$. Of course, this cannot be a proof of the total absence of physical excitations with $m^2 = -1$ on the tachyon vacuum.

²⁰This $\varpi(k)$ is not necessarily non-negative, and this may be a problem for the hermiticity of \tilde{u}_i related to u_i by (3.46) containing $\sqrt{\varpi(k)}$. For example, in the simplest case of $L = R = 1$, we have $\omega_{1A,2A}(k) \equiv 1$ and hence $\varpi(k)$ is negative for $k^2 < 1$. Though the hermiticity of the original BV states u_i itself is a problem as we mentioned in section 3.5, one way to resolve the negative ϖ problem would be to Wick-rotate to the Euclidean space-time where we have $k^2 \geq 0$ (the negative ϖ region, $0 \leq k^2 < 1$, should be regarded as an artifact of the tachyon).

6 Summary and discussions

In this paper, we carried out the analysis of the six tachyon BV states for the 2-brane solution and for the tachyon vacuum solution in CSFT. This set of six states was chosen from the requirement that the EOM of the solution holds against the states and their commutators. We found that the matrix ω_{ij} defining the BV equation is degenerate and therefore the tachyon mode is physical for the 2-brane solution. On the other hand, ω_{ij} is non-degenerate on the tachyon vacuum solution, implying that the candidate tachyon field is in fact unphysical there. These results are in agreement with our expectation and the general proof of the non-existence of physical excitations on the tachyon vacuum [8].

Our analysis in this paper is incomplete in several respects. First, we have not identified all of the four tachyon fields of the same $m^2 = -1$ which should exist on the 2-brane solution. Secondly and more importantly, we must resolve the problem that our six tachyon BV states (3.13) do not satisfy the hermiticity condition (3.56). Even if we put aside this problem, there are a number of questions to be understood concerning our tachyon BV states:

- The construction of our tachyon BV states (3.13) is not a unique one. In particular, the division of $i\mathcal{Q}u_0$ into u_{1A} and u_{1B} and that of $\mathcal{Q}u_{1A/B}$ into u_{2A} and u_{2B} (see (3.16)) have much arbitrariness which is not reduced to a linear recombination among the two states. We have to confirm that the (non-)existence of physical tachyon fluctuation does not depend on the choice of the tachyon BV states so long as they satisfy the EOM conditions. (Or we have to establish a criterion for selecting a particular set of the BV states besides the EOM conditions.)

In this paper, we introduced one parameter ξ representing an arbitrariness of the tachyon BV states (recall (3.13)). For the 2-brane solution and for $(m, n) = (1, 1)$ and $v_{1/R}$ satisfying (4.36) from the EOM conditions, we found in section 4.3 that the matrix $\omega_{ij}(k)$ is totally independent of the parameter ξ , implying that a physical tachyon fluctuation exists for any ξ . For the tachyon vacuum solution and for $(m, n) = (0, 0)$, the results for $\omega_{0,3}$ given in (5.4) and (5.7) are independent of ξ . Though we have to evaluate other ω_{ij} for confirming the non-degeneracy of the 2×2 part Ω (3.27) with $\det \Omega = \omega_{0,3}\varpi$, the fact that $\omega_{0,3}(k) \neq 0$ supports that the present set of the tachyon BV states is an unphysical one for any ξ .

Besides the analysis presented in sections 4 and 5, we carried out the analysis also for the BV states (3.13) using another choice of \hat{u}_i given by (3.18). The results for this BV states are mostly the same as those for the BV states using \hat{u}_i of (3.14). First, for the 2-brane solution, the EOM conditions are all satisfied for $(m, n) = (1, 1)$ and $v_{1/R}$ satisfying (4.36), and we obtain, in the particular case of $\xi = 1$,

$$\omega_{0,3} = 0, \quad \omega_{1A,2A} = -1 + O(k^2 - 1), \quad \omega_{1A,2B} = 0, \quad (6.1)$$

where $\omega_{1A,2A}$ is given by the following $W_{1A,2A}^{(a,b)}$:

$$W_{1A,2A}^{(a,b)} = f(s_a, s_b, \tilde{s}_a, \tilde{s}_b) - \left(\frac{\pi}{2} + s_a + \tilde{s}_b\right)^{2(1-k^2)} - \left(\frac{\pi}{2} + s_b + \tilde{s}_a\right)^{2(1-k^2)}, \quad (6.2)$$

with f defined by (4.42). This result should be compared with (4.45) for the choice (3.14) of \hat{u}_i adopted in section 4. Eq. (6.1) implies that ω_{ij} is degenerate and the tachyon is physical. However, the fact that $\omega_{1A,2A} = -1$ at $k^2 = 1$ implies that the tachyon field kinetic term given by (4.49) has the wrong sign, namely, that the physical tachyon is a negative norm one. Of course, we have to resolve the hermiticity problem before taking this problem seriously. Secondly, for the tachyon vacuum solution and for $(m, n) = (0, 0)$ and $\xi = 1$, we found that ω_{ij} is non-degenerate and hence the fluctuation is an unphysical one. The main difference from the case of \hat{u}_i given by (3.14) is that ω_{ij} are of $O(1/\varepsilon)$; for example, $\omega_{0,3} = -(\pi/2)^{2(1-k^2)} [1 + 6/(\pi^3(2k^2 - 1)\varepsilon)]$ for $L = R = 1$.

These two results, one concerning the parameter ξ and the other for another choice (3.18) of \hat{u}_i , may support the expectation that the (un)physicalness of the tachyon fluctuation is insensitive to the details of the choice of the BV states. In any case, we need a deeper understanding and general proof of this expectation.

- We have restricted our analysis of the kinetic term \mathcal{S}_0 (3.12) only to the six tachyon BV states and ignored the presence of all other states. For this analysis to be truly justified, we have to show that the complete set of the BV states of fluctuation can be constructed by adding to our set of tachyon BV states its complementary set of BV states which are orthogonal (in the sense of $\omega_{ij} = 0$) to the former set.
- As (m, n) specifying the leading small K_ε behavior of $L(K_\varepsilon)$ and $R(K_\varepsilon)$, we have considered only the two cases, (1, 1) and (0, 0). We should examine whether there are other allowed (m, n) passing the EOM tests, and if so, we must clarify the relationship among the BV bases with different (m, n) .

Finally, we have to extend our analysis to more generic n -brane solutions (including the exotic one with $n = -1$), and also to fluctuations other than the tachyon mode.

Acknowledgments

This work was supported in part by a Grant-in-Aid for Scientific Research (C) No. 25400253 from JSPS.

A KBc algebra and correlators

Here, we summarize the KBc algebra and the correlators which we used in the text. The elements of the KBc algebra satisfy

$$[B, K] = 0, \quad \{B, c\} = 1, \quad B^2 = c^2 = 0, \quad (\text{A.1})$$

and

$$Q_B B = K, \quad Q_B K = 0, \quad Q_B c = cKc. \quad (\text{A.2})$$

Their ghost numbers are

$$N_{\text{gh}}(K) = 0, \quad N_{\text{gh}}(B) = -1, \quad N_{\text{gh}}(c) = 1. \quad (\text{A.3})$$

In the text and in appendix B, there appear the following CSFT integrations:

$$\int B c e^{-t_1 K} c e^{-t_2 K} c e^{-t_3 K} c e^{-t_4 K} = \langle B c(0) c(t_1) c(t_1 + t_2) c(t_1 + t_2 + t_3) \rangle_{t_1+t_2+t_3+t_4}, \quad (\text{A.4})$$

$$\int c e^{-t_1 K} c e^{-t_2 K} c e^{-t_3 K} = \langle c(0) c(t_1) c(t_1 + t_2) \rangle_{t_1+t_2+t_3}. \quad (\text{A.5})$$

They are given in terms of the correlators on the cylinder with infinite length and the circumference ℓ :

$$\begin{aligned} \langle B c(z_1) c(z_2) c(z_3) c(z_4) \rangle_\ell = & \left(\frac{\ell}{\pi} \right)^2 \left\{ -\frac{z_1}{\pi} \sin \left[\frac{\pi}{\ell} (z_2 - z_3) \right] \sin \left[\frac{\pi}{\ell} (z_2 - z_4) \right] \sin \left[\frac{\pi}{\ell} (z_3 - z_4) \right] \right. \\ & + \frac{z_2}{\pi} \sin \left[\frac{\pi}{\ell} (z_1 - z_3) \right] \sin \left[\frac{\pi}{\ell} (z_1 - z_4) \right] \sin \left[\frac{\pi}{\ell} (z_3 - z_4) \right] \\ & - \frac{z_3}{\pi} \sin \left[\frac{\pi}{\ell} (z_1 - z_2) \right] \sin \left[\frac{\pi}{\ell} (z_1 - z_4) \right] \sin \left[\frac{\pi}{\ell} (z_2 - z_4) \right] \\ & \left. + \frac{z_4}{\pi} \sin \left[\frac{\pi}{\ell} (z_1 - z_2) \right] \sin \left[\frac{\pi}{\ell} (z_1 - z_3) \right] \sin \left[\frac{\pi}{\ell} (z_2 - z_3) \right] \right\}, \quad (\text{A.6}) \end{aligned}$$

$$\langle c(z_1) c(z_2) c(z_3) \rangle_\ell = \left(\frac{\ell}{\pi} \right)^3 \sin \left[\frac{\pi}{\ell} (z_1 - z_2) \right] \sin \left[\frac{\pi}{\ell} (z_1 - z_3) \right] \sin \left[\frac{\pi}{\ell} (z_2 - z_3) \right]. \quad (\text{A.7})$$

Finally, the matter correlator is given by

$$\langle e^{ik \cdot X(z, \bar{z})} e^{ik' \cdot X(z', \bar{z}')} \rangle_\ell^{\text{matt}} = \left| \frac{\ell}{\pi} \sin \frac{\pi(z - z')}{\ell} \right|^{-2k^2} \times (2\pi)^{26} \delta^{26}(k + k'). \quad (\text{A.8})$$

B E_ℓ , $E_{\ell,0}^{(a,b)}$, $W_{0,3}^{(a,b)}$, $W_{1A,2A}^{(a,b)}$ and $W_{1A,2B}^{(a,b)}$

In this appendix, we present explicit expressions of the quantities defined by (3.36), (3.37) and (3.38). In these expressions, $[K_\varepsilon, G_\varepsilon]_C$ denotes the following abbreviation:

$$[K_\varepsilon, G_\varepsilon]_C = K_\varepsilon c G_\varepsilon - G_\varepsilon c K_\varepsilon. \quad (\text{B.1})$$

$E_\ell(s, \tilde{s})$ ($\ell = A, B, C$).

$$E_A = \varepsilon \int B c (1 - G_\varepsilon) K_\varepsilon^m e^{-(\alpha+s)K_\varepsilon} c \frac{e^{-(\alpha+\tilde{s})K_\varepsilon}}{G_\varepsilon K_\varepsilon^n} c [K_\varepsilon, G_\varepsilon]_C, \quad (\text{B.2})$$

$$\begin{aligned} E_B = & (1 - k^2) \varepsilon \int B c K c \frac{e^{-(\alpha+\tilde{s})K_\varepsilon}}{G_\varepsilon K_\varepsilon^n} c [K_\varepsilon, G_\varepsilon]_C (1 - G_\varepsilon) \frac{e^{-(\alpha+s)K_\varepsilon}}{K_\varepsilon^{1-m}} \\ & + \varepsilon^2 \int B c (1 - G_\varepsilon) \frac{e^{-(\alpha+s)K_\varepsilon}}{K_\varepsilon^{1-m}} c \frac{e^{-(\alpha+\tilde{s})K_\varepsilon}}{G_\varepsilon K_\varepsilon^n} c [K_\varepsilon, G_\varepsilon]_C, \quad (k^2 = 0), \quad (\text{B.3}) \end{aligned}$$

$$E_C = -\varepsilon^2 \int B c \frac{e^{-(\alpha+\tilde{s})K_\varepsilon}}{G_\varepsilon K_\varepsilon^n} c [K_\varepsilon, G_\varepsilon]_C c (1 - G_\varepsilon) K_\varepsilon^{m-1} e^{-(\alpha+s)K_\varepsilon}. \quad (\text{B.4})$$

In (B.3), we have kept k^2 to make explicit the origin of the term, though we of course have to put $k_\mu = 0$ due to momentum conservation. We have omitted the space-time volume $(2\pi)^{26} \delta^{26}(k = 0) = VT$ on the r.h.s. of (B.2)–(B.4).

$$E_{\ell,0}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b) \quad (\ell = A, B, C).$$

$$E_{A,0}^{(a,b)} = -\varepsilon \int Bc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c [K_\varepsilon, G_\varepsilon]_C \left[c, (1-G_\varepsilon) K_\varepsilon^{m_a} e^{-(\alpha+s_a)K_\varepsilon} \right] V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} \\ - \varepsilon \int Bc V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{n_b-m_a}} c V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c [K_\varepsilon, G_\varepsilon]_C (1-G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}}, \quad (\text{B.5})$$

$$E_{B,0}^{(a,b)} = -(1-k^2) \varepsilon \int Bc Kc V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} V_k \left[c, \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} \right] [K_\varepsilon, G_\varepsilon]_C (1-G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} \\ - (1-k^2) \varepsilon \int Bc Kc V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c [K_\varepsilon, G_\varepsilon]_C (1-G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}} \\ - \varepsilon^2 \int Bc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c [K_\varepsilon, G_\varepsilon]_C \left[c, (1-G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} \right] V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} \\ - \varepsilon^2 \int Bc V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}} c V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c [K_\varepsilon, G_\varepsilon]_C (1-G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}}, \quad (\text{B.6})$$

$$E_{C,0}^{(a,b)} = -\varepsilon^2 \int Bc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c [K_\varepsilon, G_\varepsilon]_C \left[c, \frac{1}{G_\varepsilon} - 1 \right] G_\varepsilon (1-G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} \\ - \varepsilon^2 \int Bc V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon c \frac{K_\varepsilon}{G_\varepsilon} c G_\varepsilon (1-G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}} \\ + \varepsilon^2 \int Bc V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon (1-G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} V_k \left[c, \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} \right] \\ \times [K_\varepsilon, G_\varepsilon]_C (1-G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} + [\text{the last term with } (a) \leftrightarrow (b)]. \quad (\text{B.7})$$

$$W_{0,3}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b).$$

$$W_{0,3}^{(a,b)} = -\varepsilon \int Bc (1-G_\varepsilon) K_\varepsilon^{m_b} e^{-(\alpha+s_b)K_\varepsilon} c Kc V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}} V_{-k} \left[c, \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} \right] G_\varepsilon \\ - \varepsilon \int Bc V_{-k} K_\varepsilon^{m_b-n_a} e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon} c Kc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon (1-G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}}. \quad (\text{B.8})$$

$$W_{1A,2A}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b).$$

$$W_{1A,2A}^{(a,b)} = W_{1A,2A}^{(a,b)(1)} + \xi_a W_{1A,2A}^{(a,b)(2)}, \quad (\text{B.9})$$

with

$$W_{1A,2A}^{(a,b)(1)} = \int c V_{-k} K_\varepsilon^{m_b-n_a} e^{-(2\alpha+\tilde{s}_a+s_b)K_\varepsilon} c Kc V_k K_\varepsilon^{m_a-n_b} e^{-(2\alpha+\tilde{s}_b+s_a)K_\varepsilon}, \quad (\text{B.10})$$

$$W_{1A,2A}^{(a,b)(2)} = \varepsilon \int Bc V_{-k} K_\varepsilon^{m_b-n_a} e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon} c Kc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon (1-G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} \\ + \varepsilon \int Bc (1-G_\varepsilon) K_\varepsilon^{m_b} e^{-(\alpha+s_b)K_\varepsilon} c Kc V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}} V_{-k} \left[c, \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} \right] G_\varepsilon. \quad (\text{B.11})$$

$$W_{1A,2B}^{(a,b)}(s_a, s_b, \tilde{s}_a, \tilde{s}_b).$$

$$W_{1A,2B}^{(a,b)} = (1 - \xi_b) W_{1A,2B}^{(a,b)(1)} + \xi_b W_{1A,2B}^{(a,b)(2)} + \xi_a (1 - \xi_b) W_{1A,2B}^{(a,b)(3)} + \xi_a \xi_b W_{1A,2B}^{(a,b)(4)}, \quad (\text{B.12})$$

with

$$W_{1A,2B}^{(a,b)(1)} = \varepsilon \int Bc (1 - G_\varepsilon) K_\varepsilon^{m_a} e^{-(\alpha+s_a)K_\varepsilon} c V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{n_a-m_b}} c V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon + [(a) \rightleftharpoons (b)], \quad (\text{B.13})$$

$$\begin{aligned} W_{1A,2B}^{(a,b)(2)} = & (1 - k^2) \left\{ \varepsilon \int Bc Kc V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{n_b-m_a}} c V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon (1 - G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} \right. \\ & + \varepsilon \int Bc Kc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon \left[(1 - G_\varepsilon) K_\varepsilon^{m_a} e^{-(\alpha+s_a)K_\varepsilon}, c \right] V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} \left. \right\} \\ & + \varepsilon^2 \int Bc (1 - G_\varepsilon) K_\varepsilon^{m_a} e^{-(\alpha+s_a)K_\varepsilon} c V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} c V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon \\ & + \varepsilon^2 \int Bc (1 - G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} c V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{n_b-m_a}} c V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon, \quad (\text{B.14}) \end{aligned}$$

$$\begin{aligned} W_{1A,2B}^{(a,b)(3)} = & -\varepsilon^2 \int Bc V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{n_a-m_b}} c V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon c (1 - G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} \\ & + \varepsilon^2 \int Bc V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon c (1 - G_\varepsilon) K_\varepsilon^{m_b} e^{-(\alpha+s_b)K_\varepsilon} c V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}}, \quad (\text{B.15}) \end{aligned}$$

$$\begin{aligned} W_{1A,2B}^{(a,b)(4)} = & (1 - k^2) \left\{ -\varepsilon^2 \int Bc Kc V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon c (1 - G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} \right. \\ & + \varepsilon^2 \int Bc V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon c (1 - G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} [K, c] V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}} \left. \right\} \\ & - \varepsilon^3 \int Bc V_{-k} \frac{e^{-(2\alpha+s_b+\tilde{s}_a)K_\varepsilon}}{K_\varepsilon^{1+n_a-m_b}} c V_k \frac{e^{-(\alpha+\tilde{s}_b)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_b}} c G_\varepsilon c (1 - G_\varepsilon) \frac{e^{-(\alpha+s_a)K_\varepsilon}}{K_\varepsilon^{1-m_a}} \\ & + \varepsilon^3 \int Bc V_{-k} \frac{e^{-(\alpha+\tilde{s}_a)K_\varepsilon}}{G_\varepsilon K_\varepsilon^{n_a}} c G_\varepsilon c (1 - G_\varepsilon) \frac{e^{-(\alpha+s_b)K_\varepsilon}}{K_\varepsilon^{1-m_b}} c V_k \frac{e^{-(2\alpha+s_a+\tilde{s}_b)K_\varepsilon}}{K_\varepsilon^{1+n_b-m_a}}. \quad (\text{B.16}) \end{aligned}$$

C Seven integration regions for three $1/K_\varepsilon$

In section 4.2.1, we explained how to evaluate correlators with two $1/K_\varepsilon$ by dividing the integration region of the corresponding Schwinger parameters into three subregions (4.17). Here, we extend this to the case of three $1/K_\varepsilon$ with the corresponding Schwinger parameters (t_1, t_2, t_3) .

First, we parametrize (t_1, t_2, t_3) in terms of another set of variables (u, x, p) as

$$t_1 = \frac{u}{\varepsilon} x, \quad t_2 = \frac{u}{\varepsilon} (1 - x) p, \quad t_3 = \frac{u}{\varepsilon} (1 - x) (1 - p), \quad (\text{C.1})$$

which satisfies $t_1 + t_2 + t_3 = u/\varepsilon$. The integration range of (u, x, p) is $0 \leq u < \infty$, $0 \leq x, p \leq 1$. For a correlator multiplied by a positive power of ε , we have only to consider the integration regions where at least one of the three t_i are large, and, in the present case, there are seven such regions shown in table 1.

Region	t_1	t_2	t_3
IA	finite	finite	∞
IB	finite	∞	∞
IC	finite	∞	finite
IIA	∞	finite	∞
IIB	∞	∞	∞
IIC	∞	∞	finite
III	∞	finite	finite

Table 1. Seven integration regions.

In each of the seven regions, we adopt the following set of three integration variables:

$$\begin{aligned}
 \text{IA : } & (u, y, z) \quad \text{with} \quad x = (\varepsilon/u) y, \quad p = (\varepsilon/u) z, \\
 \text{IB : } & (u, y, p) \quad \text{with} \quad x = (\varepsilon/u) y, \\
 \text{IC : } & (u, y, z) \quad \text{with} \quad x = (\varepsilon/u) y, \quad 1 - p = (\varepsilon/u) z, \\
 \text{IIA : } & (u, x, z) \quad \text{with} \quad p = (\varepsilon/u) z, \\
 \text{IIB : } & (u, x, p), \\
 \text{IIC : } & (u, x, z) \quad \text{with} \quad 1 - p = (\varepsilon/u) z, \\
 \text{III : } & (u, y, p) \quad \text{with} \quad 1 - x = (\varepsilon/u) y.
 \end{aligned} \tag{C.2}$$

In each region, we Laurent-expand the integrand with respect to ε by regarding the specified integration variables kept fixed. The integration ranges, $[0, 1]$ for x and p , and $[0, u/\varepsilon]$ for y and z , should be appropriately modified to avoid overlaps among the seven regions as given in (4.19)–(4.21) for x and y . Finally, the u -integration should be carried out in the range $u > \varepsilon$ as given there.

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